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The Bakhshâlî Mathematics.

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Introductory

In 1881, at Bakhshâlî, a village near the city of Peshawar in the north-western corner of India, was discovered in course of excavation by a farmer, a manuscript of a work on mathematics, written on birch-bark "The greater portion" of the manuscript is destroyed and the remains consist of some 70 leaves of birch-bark but some of these are mere scraps Hoernle published two accounts of it, a short one[1] in 1883 and a fuller account[2] in 1886. This last description was republished, with some additions in 1888[3] The work has recently been printed and published by the Government of India with photographic facsimiles and transliteration of the text together with a very comprehensive introduction by Mr. G. R. Kaye.[4]

The Bakhshâlî work is a compendium of rules and illustrative examples, together with their solutions It is devoted to Arithmetic, Algebra and Geometry (including Mensuration). But comparatively very few problems dealing with Geometry have remained in the sur-

[1] *Indian Antiquary*, xii (1883), pp 89 90

[2] Verhandlungen des vii Internationalen Orientalisten Congresses, Arische Section, pp 127 *et seq*

[3] *Ind Ant*, xvii (1888), pp 33-48, 275-9

[4] *The Bakhshâlî Manuscript*—A study in Mediaeval Mathematics - Parts I and II, Calcutta, 1927, hereafter this book will be referred to as *Bakh Ms* Part III of the work is still to be out Kaye made two previous communications on the subject-matter of the Bakhshâlî work (1) "Notes on Indian Mathematics—Arithmetical Notation" (*Journ Asiat Soc Beng*, III, 1907), and (ii) "The Bakhshâlî Manuscript," *Ibid*, VIII, (1912),

viving portion of the work, the major part of which is only of arithmetical interest The topics of discussion are found to include fraction, square-root, arithmetical and geometrical progressions, income and expenditure, profit and loss, computation of gold, interests, rule of three, summation of complex series, simple equation, simultaneous linear equations, quadratic equation, indeterminate equation of the second degree, mensuration and miscellaneous problems The treatment of all these subjects is found commonly in other Hindu treatises on mathematics. More than this we are not in a position now to define the scope of the Bakhshâlî work It should, however, be noted that the sections dealing with those various topics are not well-defined Rules and examples pertaining to any one subject are oftentimes found mixed up with those pertaining to another One feature of the Bakhshâlî work deserves more notice than anything else: "Although the work is arithmetical in form it would be misleading to describe it as a simple arithmetical text-book. No algebraical symbolism is employed, but the solutions are often given in such a general form as to imply the complete general solution, *i.e*, the solutions, though arithmetical in form, are really generalised arithmetic, or algebra."[1]

Various opinions about its age.

The composition of the original Bakhshâlî work has been referred to various dates by the previous scholars. Hoernle says "I am disposed to believe that the composition of the former (the Bakhshâlî work) must be referred to the earliest centuries of our era, and that it may date from the third or fourth century A.D."[2] This estimation about the age of the original Bakhshâlî work has been accepted as fair by eminent orientalists like Buhler[3] and historians of mathematics like Cantor [4] and Cajori.[5] Thibaut would like to put it as indefinite but he has, however, followed Hoernle in accepting the date of the present manuscript to be lying between 700 and 900 A.D.[6] But Kaye would refer the work to a period about the twelfth

1 *Bakh Ms*, § 38
2 *Ind Ant*, xvii, p 36
3 *Indian Paleography*, p 82
4 M Cantor, *Geschichte der Math* I, p 598.
5 F Cajori, *History of Mathematics*, 2nd ed, Boston, 1922, p 85.
6 G Thibaut, *Astronomie, astrologie und mathematik*, p. 75

century "The script, the language, the contents of the work," says he, "as far as they can give any chronological evidence, all point to about this period, and there is no evidence whatever incompatible with it."[1] We believe, with Hoernle, that the work was written towards the beginning of the Christian era. Our arguments in support of this view will be given later on

Bakhshâlî mathematics older than the present manuscript

Hoernle thinks that the mathematical treatise contained in the Bakhshâlî manuscript is considerably older than the present manuscript itself. "Quite distinct from the question of the age of the manuscript," says he, "is that of the work contained in it There is every reason to believe that the Bakhshâlî arithmetic is of a very earlier date than the manuscript in which it has come down to us "[2] This conclusion has been disputed and rejected by Kaye who thinks it to be based on unsatisfactory grounds He then adds, "Of course it will be impossible to say definitely that the manuscript is the original and only copy of the work but we shall be able to show that there is no good reason for estimating the age of the work as different from the age of the manuscript to any considerable degree."[3] Kaye has adversely criticised the linguistic and palæographic evidence of Hoernle I frankly confess that I am as ignorant of the abstruse sciences of language and of palæography as any other layman. So I am not in a position to judge the comparative value of the evidence and arguments advanced by the either sides in respect of those matters of the controversy But what I feel is this Kaye's arguments, if proved sound and sufficient, will establish at the most that the present manuscript was written about the twelfth century, as is contended

[1] *Bakh. Ms*, § 135

[2] *Ind Ant*, xvii, p. 36

[3] *Bakh Ms*, § 122.

In support of this opinion, Kaye states "There is evidence that the Ms. is not a copy at all It is not the work of a single scribe there are cross references to leaves of the manuscript', there is a case of wrongly numbering a *sūtra* and the mistake is noted in another hand-writing" (p 74 fn) The facts noted in the latter part of this statement cannot possibly support what is stated in the beginning. On the contrary they strongly tend to show that the present manuscript is a copy.

by him[1] Hoernle himself considers it to be not much older, belonging probably to a period about the ninth century of the Christian era.[2] Most of the other reasons of Kaye against Hoernle's view, based on certain internal evidence, such as (1) the general use of the decimal place-value notation, (2) the occurrence of the approximate square-root rule and (3) the employment of the *regula falsi*, will be shown to be resting on imperfect knowledge of the scope and development of Hindu mathematics There is, however, other internal evidence of unquestionable value to show that the Bakhshâlî mathematics cannot belong to so late a period in which Kaye would like to place it.

Bakhshâlî work a commentary.

There is another noteworthy fact about the work contained in the present Bakhshâlî manuscript. From the method of its treatment Hoernle thinks it to be a *karana* work[3] I am led still further to the conviction that the Bakhshâlî work is not a treatise on mathematics in its true sense, but a commentary—a running commentary, of course,—on such an earlier work The manner of its composition and particularly the very elaborate, rather over-elaborated details with which the various workings of the solution are most carefully recorded, without trying to avoid even unnecessary repetitions, strongly tend to such a conclusion Here and there are given explanatory notes of passages, literary synonyms of words and technical terms, some of which will noway be considered difficult, or which are already well-established For instance, on folio 3 verso, the word *parasparakrtam* has been explained by the word *gunitam* (*tata parasparakrtam gunitam*), again on a subsequent occasion, this latter term has been interpreted as equivalent to another more difficult and less known term *abhyāsa* (*tatra guna abhyāsam*, folio 27, recto). On another occasion we have *āvrttipravrttigunam* (folio 12, recto)[4]

[1] *Bakh Ms*, § 135.

[2] *Ind Ant*, xvii, p 36

[3] *Ind Ant*, xii, p. 89

[4] For a different use of the term *pravrtti*, see folio 14, verso, and 15.

It is stated in several instances that *kṣayamapāstam*[1] Here is a very typical passage from the work[2] —

> " ... *dvighnamādi* | *ādidviguna* | 2 | *chayojjhitam* | *cha (ya) uttaram* | *ato uttaram pātayitvā ekam bhavati* "

The above style of composition is very characteristic of a commentary. And the whole work is written more or less in the same style.

Happily we have in the work still more conclusive proof of this There are some cross-references which are of immense help. For instance it has been observed about the 10th *sūtra* (rule), which refers to a method of multiplication, that " this rule is explained on the second page."[3] A similar remark has been made about the 14th *sūtra* that it is " written on the seventh page "[4] The importance of these two observations in determining the character of the Bakhshâlî work cannot be over-estimated[5] It will be easily recognised that those observations cannot in any way be due to the author of the original treatise For evidently those *sūtras* occur at two places in the work. No author is likely to retain consciously such recurrences in his work and pass them over merely by giving a cross-reference So the duplications, as also the observations, must be attributed to a second person, the commentator. And they happened in this way · The original Bakhshâlî treatise was not a systematic work It was an ordinary compendium of mathematical rules and examples in which the rules relating to the same topic of discussion even, were not always put together at the same place. We notice this irregularity of treatment to a certain extent in the portion of the treatise which has been left to us. It may be pointed out that such irregular treatment is not at all unusual in case of early works and we find another instance of the kind in the *Āryabhatīya* of Āryabhata (499 A. D). The commentator very properly attempted to improve upon the order, rather disorder, of the author, here and there, as far as possible, without disturbing it too much, by noticing

[1] Folios 10, 12, 14.

[2] Folio 65, verso The portion within the brackets ( ) has been restored

[3] *evaṁ sūtraṁ dvitīya patre vivaritāsti*, folio 1, recto

[4] *(sa)ptam(aṁ) patrebhilikhitasti*, folio 3, recto The letters within the brackets are missing in the manuscript

[5] For similar other remarks see folio 4 (verso)

and commenting upon at the same time the *sūtras* which are very closely connected. So that he had sometimes to explain a *sūtra* earlier than its turn according to the plan of the author Sometimes a commentator is compelled to refer to a subsequent *sūtra* before time owing to indiscretion of the author [1] Therefore when there comes the proper turn for the explanation of such a *sūtra*, he simply passes it over, very naturally, by giving the cross-reference to previous pages Thus there will remain very little doubt that the present Bakhshâlî work is a running commentary on an earlier work Further there are found other cross-references which very strongly suggest that the illustrative examples are also due to the original author [2]

Present manuscript a copy

Inspite of what is stated on the contrary by Kaye,[3] there are many things to make one believe that the present manuscript is not the original of the Bakhshâlî work, but is a copy from another manuscript For it exhibits writings of more than one scribe, possibly of five [4] This can be explained most satisfactorily only on the assumption that it is a copy Further on folio 4, verso, is found an observation as regards a certain *sūtra* (rule) that "there is mistake in the rule" (*sūtre bhrāntirasti*) The style of writing of this observation is same as that of other writings on the leaf So there is absolutely no doubt that all the writings on the leaf are due to the same scribe. Moreover, though this observation is placed between two lines of writings, it is not an ordinary case of interlining. From the apportionment of space in and about the remark, it is apparent that the remark was introduced at the time of making the copy, but not on any subsequent occasion. Now that observation cannot be due to the

[1] For instance, the author may give an illustrative example which may involve a mathematical principle which is yet to be explained An instance of the kind is found in the *Triśatikā* of Śrīdhara where the author very indiscretely gives two examples (Ex 7) in illustration of the Rule 19, which involves mathematical principles explained in the Rules 23 and 24 In this work the commentator, who is no other than the author of the treatise himself, gives the cross-references (*Triśatikā*, p 7)

[2] *Vide infra*, p 9 footnote 5

[3] *Bakh Ms*, p 74 footnote

[4] *Ibid*, pp 11, 97.

author of the original treatise. For no author would pass over a mistake in his work with a mere observation that it is wrong. So it must be from another person, possibly the scribe There is also author possibility, and there are reasons to believe it to be more probable, that the scribe found it in the copy which he used. In any case, it will follow that the present manuscript is a copy. A more conclusive proof of this is furnished by the colophon that the work is "written (*likhitam*) by a Brāhmana mathematician, son of Chajaka, for the education of the son of Vaśista"[1] Had Chajaka been the another of the work, the more appropriate and *usual* word for this colophon to begin with would have been *krtam* or *viracitam* ("composed")

The scribe seems to be a careless one For the manuscript is full of slips and mistakes Here are a few of them .—

(1) On folio 4, verso, occurs the passage "shoḍaśamasūtram 17" Evidently the figure should be 16.

(2) On folio 8, recto, a portion "uttarārdhenabhājayet" is deleted This was written by mistake for "uttarenabhajet" which is the relevant partion of the *sūtra* meant for quotation there. The deleted portion can be traced to a preceding *sūtra* (folio 7, verso).

(3) On folio 11, verso, $\begin{array}{c}158\\1\\5\\1\\64\end{array}$ is twice miswritten for $\begin{array}{c}158.\\13\\64\end{array}$ This latter fraction is once again wrongly written as "158 to 1 śe $\left|\begin{array}{c}1\\64\end{array}\right|$." Another mistake on the leaf is "93 to. ūśa 9," what is meant $93\begin{smallmatrix}1\\9\end{smallmatrix}$.

(4) In the Bakhshâlî manuscript, the end of a *sūtra* is usually marked by Owing to the carelessness of the scribe, the sign has been put many times at an intermediate place in the *sūtra*.[2]

These are considered sufficient to show the carelessness of the scribe.

[1] Folio 50, verso · *vaśistaputraha śikasyārthe putra pautra upayogyaṁ bhavatuḥ likhitaṁ chajakaputra ganakarāja brahmaṇena.*

[2] For instance see folio 4, verso, 5, recto, 8, verso, 10, recto, 16 verso, etc.

Distinguishing features of the Bakhshâlî mathematics.

We thus notice in the present Bakhshâlî manuscript the handiwork of three different types of scholars (1) the writer of an original treatise, (2) the commentator, and (3) the scribe Of the latter type again there are traces of the work of no less than five different persons who by their co-operation produced the present copy We shall, however, leave these speculative and controversial matters now for broader and surer facts in respect of which there will be less scope for play of imagination and diversity of opinions. There are certain characteristic features of the Bakhshâlî mathematics, in the scope of topics discussed in it, in the method of their treatment, in the matter of symbols and notations, and last but not the least, in peculiarities of terminology, all of which considerably distinguish the work from the rest of the Hindu treatises on mathematics which are more commonly known, *e g*, the works of Āryabhata (449 A D.), Brahmagupta (628 A.D.), Śrīdhara (*c* 750 A.D.), Mahāvīra (850 A.D) and Bhāskara (1150 A.D.), as also the commentary of Prithudakasvâmî (860 A D.) on the mathematics of Brahmagupta A careful scrutiny of these characteristics, specially with a comparative view, will not only help us to make as fair an estimate as possible of the value of that work but will also be of much use in fixing closer limits to the period of its composition, about which, we have already seen, there exist widely varying opinions Hence such a study will be of much more value for the history of early Hindu mathematics than anything else.

Method of exposition

The most distinguishing feature of the Bakhshâlî work and the one which strikes the mind of its readers first of all, is its too elaborate method of exposition which in certain respects is characteristically its own. A rule, called *sūtra*, is stated first. It is then illustrated by a few examples, called *udāharana*, which is in most places abbreviated into *udā*. Sometimes the example is called *praśna* ("question").[1] Each example is followed by a formal statement of the problem in terms of numerical figures and words or abbreviations indicative of

[1] Folios 46, verso, and 65, recto

symbols of operation and of other relevant matters. It is generally called *sthāpana*, but on occasions *nyāsa* [1] or *nyāsa-sthāpana* [2] After this comes the careful record of the very elaborate details of the workings of the solution of the example, called *karana*, in course of which are oftentimes quoted fragments of the *sūtra* under which the example is placed If a solution requires the help of another *sūtra*, that is also quoted For instance, the rule for finding the approximate value of a surd is found to have been quoted again and again on every occasion where it is applied This method is now of great help to us not only to restore some of the mutilated *sūtras* but also to reclaim others which have been completely destroyed in the present remains of the Bakhshâlî work It is, in fact, from quotations in this way that we have come to know of the existence of the approximate square-root rule (*vide infra*) Finally comes the verification of the solution, called *pratyaya*. Sometimes the same solution is verified in more than one way [3]

The above will explain in general the method of exposition of the Bakhshâlî work But there are also occasional deviations from it. For it is not always that a *sūtra* is illustrated by examples and an example is followed by its solution There are at least two *sūtras* in the surviving portion of the Bakhshâlî work which have no examples attached to them They have been passed over as having been explained or written on preceding pages Two examples are left without solution with similar remarks. Again solutions of examples

[1] Folios 23, recto, 25, verso, 29, recto, and 55, verso Compare also folio 35, recto (*a*) and verso (*b*)

[2] Folios 32, 36, 44, verso and 46, recto These references must have escaped the notice of Hoernle who remarks otherwise (*Ind Ant* xii, p 89, xvii, p 34).

[3] For example, on folio 11, verso, there is mention of verification by the fourth method (*anyaṁ caturtha-pratyayaṁ kriyante*)

[4] Folios 1, recto and 3, recto

[5] *Vide* folio 4, recto "(Examples) of this kind are also written previously on the eleventh page" (*evaṁ ekādaśamapatrebhilikhitā pūrvepi*) and folio 60, recto, *ekonoviṁśatima patre vivaritāsti* ("explained on the 21st page") These remarks imply that the mathematical principles involved in the examples have been stated and illustrated at different places of the Bakhshâlî work So these are further evidence of what we have already pointed out that the plan of treatment in the original Bakhshâlî treatise is not a systematic one It also strongly suggests that the examples (*udāharāna*) are also due to the original author Hence the commentator is responsible only for the statement, solution and verification of an example

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m] ly m l] by l t mm ta
(860 A D) wl l]]l l fl t mb f
tl l t d l l M l
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f l 18 l f li 28 25

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pl l b l d d tl ti C l l l
l d lg b l] tl f B l g pt S
Alg b tl A tl t l M t f tl S
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ā ñ g l li l tl P gl l t l tl by
2
lā dit l by S ll k l l B 1800
t l d by S ll k D i l l 1010
D l li d l by M llll... Jl..., B...,

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$\sqrt{18}$

21		21	
20	100	20	
		21	

57

...uare of the last term of it will be equal to the given quantity...

There are no less than five applications of the above rule in the present Bakhshâlî work, *viz.*,[1]

$$(i)\quad \sqrt{41} = 6+\frac{5}{12} - \frac{(5/12)^2}{2(6+5/12)},$$

$$(ii)\quad \sqrt{105} = 10+\frac{1}{4} - \frac{(1/4)^2}{2(10+1/4)},$$

$$(iii)\quad \sqrt{481} = 20+\frac{20}{21} - \frac{(20/21)^2}{2(21+20/21)},$$

$$(iv)\quad \sqrt{889} = 29+\frac{24}{29},$$

$$(v)\quad \sqrt{339{,}009} = 579+\frac{384}{579} - \frac{(384/579)^2}{2(579+384/579)}$$

The above approximate formula is now generally attributed to the Greek Heron (*c.* 200 A D.)[2], and it is restated by the Arab Al-Haṣ-sarâr (*c* 1175 A D ?) and other mediæval algebraists[3] But it was known, as has been shown elsewhere, to the second order of approximation, to the ancient Hindus several centuries before[4]

[1] Folios 57 and 64, verse, 45 recto, 56, recto and 65, verso, 45 and 46, recto Note the expression *mūlam kṣiṣṭakaraṇyā* or "the root by the method of approximation" (Folio 65, verso)

[2] T Heath, *History of Greek Mathematics*, vol ii, p 324, hereafter this book will be referred to as Heath, *Greek Mathamatics* Heron's time is uncertain He may have lived in the 3rd century A D

[3] D E Smith, *History of Mathematics*, in two volumes, 1925, vol ii, p 254, hereafter this book will be referred to as Smith, *History*

[4] Bibhutibhusan Datta, "Hindu Contribution to Mathematics," *Bulletin of the Mathematical Association, University of Allahabad* Vol I (1927 28), p 69 Hereafter referred to as *Hindu Contribution.*

Āryabhata and Brahmagupta give the formulæ[1]

$$\sqrt{a^2+r} = a+\frac{r}{2a}$$

$$\sqrt[3]{a^3+r} = a+\frac{r}{3a^2}$$

Rodet [2] holds that a process of approximation to the value of a surd was known to the authors of the *Śulba-sūtras*, the earliest of which was written *c* 800 B C

$$\sqrt{a^2+r} = a+\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}+\epsilon,$$

where

$$\epsilon=\left[\frac{1}{r}-\left\{\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}\right\}\left\{2a+\frac{r}{2a+1}\right.\right.$$

$$\left.\left.+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}\right\}\right]-2\left\{a+\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{a+a+1}\right)}\right\}$$

This is an approximation of the 4th order Putting $a=1$, $r=1$, we get

$$\sqrt{2}=1+\frac{1}{3}+\frac{1}{3\ 4}-\frac{1}{3\ 4\ 34},$$

[1] Rodet, *Leçons de Calcul d'Aryabhata*, *Brāhma sphuṭa siddhānta* XII. 7, 62, compare also J Tropfke, *Geschichte der Elementar Mathematik*, Berlin, 1921, Vol. II, p. 138

[2] L. Rodet, "Sur une méthode d'approximation des racines carrés, conne dans l'Inde antérieurment à la conquête d'Alexandre," *Bull Soc Math. d. France*, VII (1879), pp 98-102; "Sur les méthodes d'approximation chez les anciens" *Ibid*, pp 159-167.

a result well-known in the *Śulba-sūtras*.[1] This rule gives an approximation by defect whereas the previous one by excess Further this was unknown to the Greeks, but the second approximation of it was known to the Arabs [2]

We thus learn that Kaye is wrong in asserting that "the square-root rule was not used by the Hindus and was not even noticed by them until the sixteenth century"[3]

Calculation of errors and Process of reconciliation.

The Bakhshâlî mathematics exhibits an accurate method of calculating errors and an interesting process of reconciliation, the like of which are not met elsewhere They are necessiated by the application of the foregoing approximate square-root formula. There are certain examples whose solution leads to the determination of the number of terms of an arithmetical progression whose number of terms is unknown, but first term (a), common difference (d) and the sum (s) are known If the number of terms be t, then, according to the Bakhshâlî work

$$s=\left\{(t-1)\frac{d}{2}+a\right\}t, \qquad (1)$$

whence $$t=\frac{-(2a-d)+\sqrt{(2a-d)^2+8ds}}{2d}$$

The negative sign of the radical has been overlooked in the Bakhshâlî work. Putting $p=2a-d$ and $Q=(2a-d)^2+8ds$, we have

$$t=\frac{-p+\sqrt{Q}}{2d}$$

$$\left.\begin{array}{ll}\text{or} & 2dt+p=\sqrt{Q},\\ \text{also} & 2s=t^2d+pt\end{array}\right\} \qquad (2)$$

[1] Thibaut suspects that this result might have been obtained by the early Hindus by some geometrical devices It has been observed elsewhere that the result was more probably obtained by the process of continued fraction (*Hindu Contribution*)

[2] Smith, *History*, ii, p 254.

[3] *Bakh Ms*, § 69, compare also §§ 120, 134

Oftentimes in the examples given the value of $\sqrt{Q}$ does not come out in exact terms, so that a method of approximation has to be adopted Let $q_1, q_2,$ be the successive approximations to the value of $\sqrt{Q}$ and let the values of t obtained from them be $t_1, t_2, \ldots$. Neither of these values will evidently give the original quantity s when substituted in the equation (1) for the purpose of verification of the results obtained. Suppose the values of s corresponding to the values of t be s_1, s_2 Then

$$2s_1 = t_1{}^2 d + pt_1$$

or $$8ds_1 + p^2 = (2t_1 d + p)^2$$

and $$8ds + p^2 = (2td + p)^2$$

therefore $$8d(s_1 - s) = (2t_1 d + p)^2 - (2td + p)^2$$

Now $$2td_1 + p = q_1$$

Hence $$s_1 - s = \frac{q_1{}^2 - Q}{8d}$$

Since $$\sqrt{Q} = \sqrt{a^2 + r} = a + \frac{r}{2a} = q_1,$$

up to the first approximation,

we have $$q_1{}^2 - Q = \left(\frac{r}{2a}\right)^2 = \epsilon_1, \text{ say}$$

Then ϵ_1 will denote the first error. Therefore

$$s_1 - s = \frac{\epsilon_1}{8d}$$

Similarly for the second approximation, the error will be[1]

$$\epsilon_2 = \left(\frac{(r/2a)^2}{2(a + r/2a)}\right)^2$$

[1] The *Kṛtıkṣaya* probably refers to this second error.

and $$s_2 - s = \frac{\epsilon_2}{8d}.$$

We shall now refer to a specific instance, in which [1]

$$a=1, \qquad d=1, \qquad s=60$$

The detailed workings given are

$$8ds=480, \qquad 2a-d=2.1-1=1. \qquad 480+1=481$$

$$\sqrt{481}=21\frac{40}{42}=\frac{882+40}{42}=\frac{922}{42}$$

Then $$t_1=\frac{1}{2}\left(\frac{922}{42}-1\right)=\frac{880}{84}$$

Hence $$s_1=\frac{t_1(t_1+1)}{2}=\frac{880}{84}\times\frac{964}{168}=\frac{848,320}{14112}$$

and $$\frac{\epsilon_1}{8d}=\frac{1}{8}\left(\frac{40}{42}\right)^2=\frac{1600}{14112}$$

$$\therefore \quad s=s_1-\frac{\epsilon_1}{8d}=\frac{848,320}{14112}-\frac{1600}{14112}=\frac{846,720}{14112}=60$$

Again for the second approximation

$$\sqrt{481}=21\frac{20}{21}-\frac{(20/21)^2}{2(21+20/21)}=\frac{425,042-400}{19,362}=\frac{424,642}{19,362}$$

$$\therefore \quad t_2=\frac{1}{2}\left(\frac{424,642}{19,362}-1\right)=\frac{405,280}{38,724}$$

[1] Folio 65, verso and 64, recto Portions of the detail workings are not preserved in the existing manuscript But they can be easily restored

Hence $$s_2 = \frac{t_2(t_2+1)}{2} = \frac{405,280}{38,724} \times \frac{444,004}{77,448}$$

$$= \frac{179,945,941,120}{2,999,096,352}$$

$$\frac{\epsilon^2}{8d} = \frac{40^4}{8^3 \times 21^4 \times (21\frac{20}{21})^2} = \frac{160,000}{2,999,096,352}$$

$$\therefore \quad s = s_2 - \frac{\epsilon^2}{8d} = \frac{179,945,941,120 - 160,000}{2,999,096,352}$$

$$= \frac{179,945,781,120}{2,999,096,352} = 60$$

Negative sign

In the Bakhshâlî manuscript a negative quantity is denoted by a cross (+) placed after the number affected. Thus 11 7+ means 11 − 7 This is very remarkable For in the manuscripts of Prithudakasvâmî (860 A D) and later Hindu writers a dot is usually placed above the quantity for the same purpose, so that according to them 11 − 7 is denoted by 11 7 The origin of the use of a cross for the negative sign has been the subject of much conjecture Thibaut has suggested its probable connexion with the Diophantine negative sign ⋔ (reversed ψ, abbreviation for λειψις, meaning "wanting") [1] This has been accepted by Kaye.[2] But such a conjecture seems to be hardly reliable. For firstly the Greek sign for minus is not ⋔ but an arrow-head (⋀) and "it is now certain," observes Heath, "that the sign has nothing to do with ψ."[3] An arrow-head and a cross are too much different to be connected together, or too distinct to be confused for each other Secondly, the Greek symbol itself is of doubtful origin. And above all, we are not sure if it is as old as it will have to be for being the precursor of the Bakhshâlî cross For there is no manuscript of the *Arithmetica* of Diophantus which is older than the Madrid copy of the thirteenth century A.D. and again in many cases in this work, the negative quantity is indicated by writing the full Greek word for "wanting" in its different case

[1] *Ind Ant*, xvii, p 34.
[2] *Bakh Ms*, §§ 127, *JASB*, viii (1912), p 357.
[3] Heath, *Greek Mathematics* II, p 459,

endings. So we cannot be sure if Diophantus did actually use that symbol for the minus sign[1] Under such circumstances it will not be proper and safe to assume the possibility of Greek connexion for the negative symbol of the Bakhshâlî manuscript Hoernle thinks it—though he is not quite confident in this respect—to be the abbreviation *ka* of the word *kanita* or *nū* (or *nu*) of the word *nyūna*, both of which means " diminished " and both of which abbreviations, in the Brāhmī characters, would be denoted by a cross[2] This supposition has got a very notable point in its favour. In the Bakhshâlî manuscript all the other arithmetical operations are generally indicated by the abbreviations[3] (initial syllables) of the words of that import, though often the words are written in full and occasionally nothing is indicated at all. So it will be very natural to search for the origin of its negative sign in that direction In this way Hoernle's hypothesis appears to be a very probable one But its principal drawback is that neither the word *kanita* nor the word *nyūna* is found to have been used in the Bakhshâlî work in connexion with the subtractive operation. The nearest approach to that sign is that of *kṣa*, abbreviated from *ksaya*, (" decrease ") which has been used several times, indeed more than any other word indicative of substraction. The sign for *ksa*, whether in the Brāhmī characters or in the Bakhshâlî characters, differs from the simple cross (+) only in having a little flourish at the lower end of the vertical line. The flourish might have been dropped subsequently for convenient simplification.

Least Common Multiple.

The plan of reducing fractions to the lowest common denominator before adding or subtracting is known correctly to the author of the Bakhshâlî mathematics We have a few instances of its application in the work. In one instance,[4] it is required to find the sum of the fractions

$$\tfrac{2}{1},\ 1\tfrac{1}{2},\ 1\tfrac{1}{3},\ 1\tfrac{1}{4},\ 1\tfrac{1}{5}$$

[1] *Cf* Smith, *History* II, p 396

[2] *Ind Ant*, xvii, p 34

[3] It may be be noted that abbreviations of all sorts of things, mathematical as well as non-mathematical, have been freely used in the Bakhshâlî work (*vide* § 62)

[4] Folio 1, verso

They are first reduced to a common denominator (*sadrsam kryate*), so as to become

$$\frac{120}{60}, \frac{90}{60}, \frac{80}{60}, \frac{75}{60}, \frac{72}{60},$$

respectively Finally the sum is stated to be $\frac{437}{60}$

In a different instance,[1] it became necessary to add up

$$\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{3}{5},$$

It is stated that the sum, after having reduced to a common denominator (*harasāmye krte yutam*), will be $\frac{163}{60}$ On reducing the fractions

$$\frac{12}{19}, \frac{4}{7}, \frac{6}{11},$$

to a common denominator, they are stated to be respectively,[2]

$$\frac{924}{1463}, \frac{836}{1463}, \frac{789}{1463}$$

A fairly difficult case is to simplify[3]

$$\frac{13\frac{1}{3}}{3\frac{1}{8}} + \frac{13-1}{8\frac{1}{2}} + \frac{1\frac{1}{3}}{3\frac{1}{5}} + \frac{\frac{1}{2}}{1\frac{1}{2}} + \frac{1}{5\frac{1}{3}} + \frac{2\frac{1}{1}}{5} + \frac{12\frac{1}{2}}{33\frac{1}{3}}$$

The result is correctly obtained as $\frac{1807}{240}$

[1] Folio 17, recto [2] Folio 2, verso.

[3] Folios 43, (recto and verso), and 44, (recto) Compare *Bakh Ms*, § 95 The manuscript is erroneous here, so is also Kaye's transliteration Our emendation is correct as it gives the correct result This is another proof of the fault of the scribe. *Vide* also folios 44 (verso) and 67 (verso)

The method of finding the least common multiple is found in the *Ganita-sāra samgraha* of Mahāvīra (c 850 A D.)[1] and probably also in Pṛithudakasvâmî's commentary on the *Brāhma-sphuta-siddhānta*,[2] but not in the works of Āryabhata, Brahmagupta and Bhāskara

Arithmetical notation—Word-numerals

The arithmetical notation generally employed throughout the Bakhshâlî work is the decimal place-value notation. This fact has been differently utilised by different writers On the one hand Hoernle[3] and Buhler,[4] who believe in the antiquity of the Bakhshâlî mathematics, consider it as evidence of the earlier date of the discovery of that notation by the Hindus On the contrary, Kaye[5] who believes in the non-Indian origin of the place-value notation and in its late introduction into India, considers its general adoption in the Bakhshâlî work as proof against the hypothesis of the previous writers about the early date of this work It is now definitely known that Kaye's notions about the origin of the place-value notation is wrong It was invented in India about the beginning of the Christian era, probably a few centuries earlier [6] But apart from that controversy it should be noted that the nearly exclusive application of this notation in the Bakhshâlî work is very much noteworthy inasmuchas in almost all the available Hindu mathematical treatises, save and except the *Āryabhaṭīya* of Āryabhata (499 A D.), we find copious use of the word-numerals. There is, however, evidence to show that the author of the Bakhshâlî work did know the principle of the word-numeral system of arithmetical notation In it we find the use of the words *rūpa* $(=1)$, *rasa* $(=6)$[7], and *pāda* $(=\frac{1}{4})$[8] with numerical

[1] *Ganita-sāra-samgraha*, iii 56 *Cf Hindu Contribution*

[2] Colebrooke, *Hindu Algebra*, p 281, footnote 1, p 289 fn Also *Brāhma-sphuṭā-siddhānta*, pp 178-9.

[3] *Ind Ant*, xvii, p. 38. [4] *Indian Paleography*, p 82 [5] *Bakh Ms*, § 131

[6] This has been proved by the writer in a series of articles "A Note on the Hindu-Arabic Numerals" (*Amer Math Month*, vol 33, 1926, pp 220 1), "Early literary Evidence of the use of the zero in India" (*Ibid*, pp 449 54), "The present mode of expressing numbers" (*Ind Hist Quart*, vol. 3, 1927, pp 530 40), "Al-Bîrûnî and the origin of the Arabic Numerals" (*Proc Benares Math. Soc.*, Vol 7, 1928)

[7] Folio 60, verso. Kaye reads it as *vasa* which is meaningless It should be *rasa* (*cf. Ind Ant.*, xvii, p 41).

[8] Folio 4, recto.

significance The use of the last word is as old as the *Vedas*. The first occurs as early as in the *Jyotisa Vedāṅga*[1] (c 1200 B C) and the second in the *Chandah-sūtra* of Pingala (before 200 B C)[2]. Again in speaking of a very large number

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the Bakhshâlî mathematics writes[3]

Sadvimśaśca tripañcāśa ekonatrimśa eva ca
Dvāsa(sti) ṣadvimsa catuhcatvālimśa saptati
Catuhsastina(va) *imśanamtaram*
Trirasīti ekavimśa asta. . *pakam*

Clearly the principle of the word-numeral system has been followed in this instance. The only departure from its popular features lies in (1) the use of the number names in the place of the word-names and (2) the adoption of the left-to-right system in the arrangement of the figures. But these features, though not common, are not altogether foreign to the system.[4] Once at least the author has followed the right-to-left sequence. For the compound word *catuh-pañca* has been used to denote once | 4 | 5 and again | 5 | 4 (folio 27, recto).

Rūponā Method.

In the Bakhshâlî mathematics, there are several mentions of an arithmetical process, called *rūponā karana*, and in every case the reference is undoubtedly to the rule for the summation of a series in arithmetical progression, *viz*

$$S=\{(t-1)\tfrac{d}{2}+a\}t.$$

[1] *Yājusa-Jyotiṣa*, 23, *Ārsa-Jyotisa*, 31

[2] *Chandaḥ sūtra*, vi, 34; viii, 2, 3, 10, 11, 18

[3] Folio 58, recto. This occurs in a problem whose only object seems to be to express this big number in figures. We do not find such a problem in any known Hindu arithmetical treatise. This bespeaks that the Bakhshâlî work must be referred to the early period of the invention of the decimal place-value notation.

[4] The writer has published a comprehensive history of the origin and development of the word numerals in the *Bangîya Sâhitya Parisad Patrika*, 1335 B S (1928), pp 8 30.

The origin of that name is supposed by Hoernle[1] and Kaye[2] to be lying in the fact that "the rule in question began with the term *rūponā* which corresponds to the $(t-1)$ of the formula." The term *rūponā* literally means, "deducting one." As the rule is not preserved in the available portion of the Bakhshâlî mathematics it is not possible to verify this supposition. Kaye, however, points out that the rule has very nearly the same beginning in the *Gaṇita-sāra-saṁgraha*[3] of Mahāvīra *Rūponena gaccho dalī kṛtaḥ* .... The above interpretation of the origin of the term *rūponā karaṇa*, though not impossible, does not appear to be very convincing. The technical terms which are commonly used in the Bakhshâlî mathematics in connexion with the arithmetical progression, such as *ādi*, *prabhava*, *caya*, *uttara pada*, *dhana*, etc, are all same as in the other Hindu treatises, the name *ruponā karana* is unique for it. It is not met with elsewhere. It is further noteworthy that no other term in the Bakhshâlî mathematics, or in any other Hindu mathematical treatise, is known to have been formed in the same way, with the opening word of the rule.

It may be noted here that in the Bakhshâlî mathematics, the word *rūpa* occurs also with different significance than unity. For instance, we find[4]

" (*bhā*) *jitam* $\left|\begin{smallmatrix}4\\2\end{smallmatrix}\right|$ *jātam* |2| *labdham sarūpa* | *eṣa rūpādhikaṁ*|3|

eṣa kāla . $\left|\begin{array}{c|c|c} 3 & u\ 4 & pa\quad 3 \\ 1 & 1 & 1 \end{array}\right|$ *rūpoṇā-karanena phalaṁ rū* 21 || *dvi-*

tīyasya trairā $\left|\begin{array}{c|c|c} 1\quad di & 7 & 3\quad di \\ 1 & 1 & 1 \end{array}\right|$ *pha rū* 21||"

or " .divided becomes 2, 'quotient plus 1 (*rūpa*),' this increased by 1 becomes 3, which time .by the *rūpoṇākaraṇa*, the result is *rū* 21. Of the second, by the rule of three, the result is *rū* 21."

In this passage the number 21 has twice been marked as *rū*, abbreviated from *rūpa* Again in a *sūtra* (folio 8, recto) related to an arithmetical progression, we find the passage *labdhaṁ rūpaṁ vinirdiśet*, that is, "the quotient should be indicated as *rūpa*" Here

[1] *Ind Ant*, xvii, p 47

[2] *Bakh Ms*, § 73

[3] ii. 63.

[4] Folio 7, verso

again the term *rūpa* seems to have a purely technical significance. There are other instances in which *rūpa* does not mean unity, but is used in connexion with an integer [1] Similar use of the word *rūpa* is found in later Hindu mathematical treatises where it denotes, besides 1, an integer or the integral part of a mixed fraction.[2] I venture to amend the word *rūponā* to *rūpana* [3] Then it will mean "making *rūpa*" which means "known or absolute number," "known quantity as having specific form" [4] So *rūpana-karaṇa* will mean "the method of making absolute number," that is, "totalisation" or "summation." This hypothesis will be strongly supported by the expression "*rūpana karanena phalam rūpa* 21" (or "by 'the method of making *rūpa*' the result is *rūpa* 21")

Symbol for the unknown

In the Bakhshâlî mathematics the unknown quantity is referred to by the symbol o, which is called *śūnya* ("void" or "empty").[5] Strictly speaking it is not a symbol for the unknown as has been supposed by Hoernle[6] and Kaye [7] For the same symbol has also been used for the "zero" (*śunya*) of the decimal arithmetical notation That is, indeed, its true significance Its use in connexion with an algebraic equation, in a sense other than for arithmetical notation, is simply to indicate that the quantity which should be there is absent or not known.[8] Hence its place in the equation is left vacant and this is clearly indicated by putting the sign of emptiness there. Or

[1] Folios 21 (recto), 60 (recto), 96 (verso) etc

[2] See *Brāhma sphuṭa siddhānta*, xii 2, *Triśatikā*, pp. 7 *et seq*, *Lālāvatī* pp 6, 7, *Bījagaṇita*, pp 2 *et seq*, (Colebrooke *Hindu Algebra*, p. 149)

[3] *Rūponā* may be an archaic form of *rūpana*.

[4] See Monier Williams, *Sanskrit English Dictionary*, revised by Cappaller and Leumann, on *rūpa* Compare this use of the word *rūpa* with its use in algebra in the sense of absolute known number in an equation.

[5] Folios 22 (verso), 23 (recto and verso), etc

[6] *Ind Ant*, xii p 90, xvii, p. 30

[7] *Journ Asiat Soc. Beng.*, viii (1912), p 357, *Bakh. Ms*, §§ 42,60

[8] Compare such expressions as *mūlaṁ na jñayate* (folio 13. verso. 15 v) *prathamaṁ na jānāmi* (24, verso), *padaṁ na jñayato* (54, verso), etc in each case of which the *ajñāta* (unknown) element has been indicated in the statement by *śūnya*.

in short, the use of the truly arithmetical symbol for zero in an algebraic equation is a clear proof of the want of a symbol for the unknown in the Bakhshâlî mathematics. Correctness of this interpretation will be borne out by the facts (1) that this symbol does nowhere enter into any operation, as it ought to have done had it been truly a symbol for the unknown, and (2) that oftentimes it is referred to as *śūnya-sthāna* or the "empty place" proving thereby that nothing is in that place.[1] This hypothesis will be further supported by the fact that the similar use of the "zero" sign to denote the unknown element in the statement (*nyāsa*) of problems is found in the *arithmetics* of Śrīdhara[2] and Bhāskara.[3] Thus we have[4]

ādiḥ 20 | *u* 0 | *gacchaḥ* 7 | *gaṇitam* 245 |

which is a statement of an arithmetical progression whose first term is 20, number of terms is 7, sum is 245 and whose common difference is not known. Both these writers have well defined notations for the unknown, and do never use the cipher in this way in their treatises on algebra. But as the use of algebraic symbols is not permissible in arithmetic, they make use of the cipher to indicate that certain element in a problem is wanting. Of course, the cipher has wider use in the Bakhshâlî mathematics than in any of these works.

The lack of an efficient symbolism is bound to give rise to a certain amount of ambiguity in the representation of an algebraic equation, especially when it contains more than one unknown.[5] For instance, in[6]

$$\left|\begin{array}{ccc|ccccc} 0 & 5\ yu\ m\bar{u} & 0 & sa & 0 & 7+ & m\bar{u} & 0 \\ 1 & 1 & 1 & & 1 & 1 & & 1 \end{array}\right|$$

[1] Folios 25 (verso) and 26 (recto).

[2] *Triśatikā*, pp. 19 et seq.

[3] *Līlāvatī*, pp. 18 et seq. This is not evident from Colebrooke's translation of the work where the cipher has been replaced by the query.

[4] *Triśatikā*, p. 29.

[5] Nearly similar difficulty and inconvenience were experienced by the Greek algebraists who had only one symbol for the unknown.

[6] Folio 59, recto. Hoernle and Kaye are not right in thinking that this statement represents

$$x+5=s^2 \text{ and } x-7=t^2$$

which denotes $\sqrt{x+5}=s$, $\sqrt{x-7}=t$, different unknowns will have to be assumed at different vacant places. Again in the statement[1]

ā	5 1	*u*	6 1	*pa*	∘ 1	*dha*	∘ 1
ā	10 1	*u*	8 1	*pa*	∘ 1	*dha*	∘ 1

which refers to two arithmetical progressions whose first terms and common differences are different but known, and whose sums and number of terms are equal but unknown, $\overset{\circ}{1}$ stands in the place of two different unknowns[2] To avoid such ambiguity, in one instance which contains as many as five unknowns, the abbreviations of ordinal numbers such as *pra* (abbreviated from *prathama*, "first"), *dvi* (from *dvitīya*, "second"), *tr* (from *tṛtīya*, "third"), *ca* (from *caturtha*, "fourth") "and *paṁ* (from *pañcama*, "fifth") have been used to represent the unknowns, *e g*[3]

9 *pra*	7 *dvi*	10 *tr*	8 *ca*	11 *paṁ*	*yutaṁ jātaṁ pratyaika (krameṇa)*
7 *dvi*	10 *tr*	8 *ca*	11 *paṁ*	9 *pra*	16 | 17 | 18 | 19 (20)

which means

$$x_1+x_2=16,\ x_2+x_3=17,\ x_3+x_4=18,\ x_4+x_5=19,\ x_5+x_1=20$$

The want of a proper symbol for the unknown eventually leads to the adoption of the method of "false position" or "supposition" for solution of algebraic equations. The solution generally begins with putting "any desired quantity" (*yadṛcchā*) in the vacant place.[4]

[1] Folio 5, recto

[2] It is not easy to say what is intended to be implied by placing the unity below the cipher. It is supposed by some to be an indication that the unknown quantity will be an integer (Kaye, *Bakh. Ms*, § 60). Such a supposition is quite untrue For in the instance cited while *dhana* is an integer (=65), *pada* is a fraction (=13/3). Strangely this very statement has been quoted by Kaye just after the remark referred to In certain instances, it is a mixed surd (*vide* folios 6 and 45, rectos)

[3] Folio 27, verso In one instance in Bhāskara's *Bījagaṇita* initial syllables of the names of particular things have been used as symbols for the unknowns. (Colebrooke, *Hindu Algebra*, p 195, compare also p, xi)

Cf. अत्र रूपाणामव्यक्तानां चाद्याक्षराण्युपलक्षणार्थं । . *Bījagaṇita*, p. 2.

[4] *Yadṛcchā pinyase śūnye* or *yadṛcchā vinyase śūnye*, that is "putting any desired quantity in the vacant place" (Folios 22, verso, and 28, recto). On another occasion it is said : *Kāmikaṃ śūnye pinyastaṃ* or "the desired quantity is placed in the vacant place" (Folio 23, recto and verso). We have also such expressions as

Origin of yāvat-tāvat for the unknown.

Later Hindu algebraists are seen to use the term *yāvat-tāvat* ("as many as" or "so much as") or its abbreviation *yā* to represent the unknown quantity in algebra We do not know when and how this term first entered into the science of algebra, but its use is found as early as in the writings of the eminent commentator and mathematician Prithudakasvâmî (860 A D)[1] This writer sometimes calls it *yāvaka* (or "as many") and still uses the abbreviation *yā*[2] Now at least from the time of Brahmagupta (628 A.D), if not earlier, the Hindus have adopted as symbols for the unknown quantities, the *varṇa*. This Sanskrit word denotes the letters of the alphabet as well as colours. And indeed both are known to have been used to represent the unknown[3] But *yāvat-tāvat* is neither an alphabet nor a colour. Hence the suspicion naturally arises how such a term came to be used for the unknown. A careful investigation into the origin of this term will most likely give a peep into the early history of the growth and development of Hindu algebra. That suspicion perhaps came to the mind of Prithudakasvâmî when he most arbitrarily and erroneously decided to call *yāvat tāvat*, a *varna*. "In an example in which there are two or more unknown quantities," says he, "two or more colours, as *yāvat tāvat*, etc. must be put for their values."[4] Bhāskara

śūnya-sthāne rupam dattvā or "putting one in the vacant place" (Folios 25, verso and 26, recto , compare also folio 22, verso) It should be noted that though the author promises to put any arbitrary quantity (*yadrcchā* or *Kāmikaṃ*) in the vacant place, in actual practice, he has in most cases put only unity. Thus we find "*yadṛcchā* ‖ 1 ‖ " and "*Kāmikam* 1 ‖ " These facts led Hoernle to conclude that these two words have probably been given in this connexion a peculiar significance as the number 'one' (*Ind Ant* , xvii, pp 39, 49) Such a conclusion has rather been too hasty For in one instance the arbitrary quantity is assumed to be 5 (*tatrecchāpañcamaḥ*, Folio 29, recto (b)), and in some other instances other values have been assumed (vide *Bakh Ms* , § 72)

[1] Colebrooke, *Hindu Algebra*, p, 344, fn 2 and p 948 fn.

It is not known now whether Brahmagupta used *yāvat-tāvat* for the unknown At least there is nothing to show that he did so The occurrence of the term in the solutions of the examples given by Brahmagupta which are found in Colebrooke's translation of the arithmetical and algebraic portions of his work cannot be taken as evidence in this respect. For, as has been already pointed out, they are not Brahmagupta's own

[2] Colebrooke, *Hindu Algebra*, p. 288, footnote 1 ; p 292 fn.

[3] *Hindu Contribution*

[4] Colebrooke, *Hindu Algebra*, p. 348 fn.

evidently could not reconcile himself with this forced interpretation of Prithudakasvâmî, so he makes distinct mention of *yāvat tāvat* and *varna* as symbols for the unknown and attribute the credit for the introduction of either symbols to the ancient mathematicians. Hence he observes ' "So much as" and the colours "black, blue, yellow, and red" and others besides these,[1] have been selected by venerable teachers for names of values of unknown quantities, for the purpose of reckoning therewith.'[2] This, however, leaves still unexplained the origin of the term to *yāvat tāvat*

According to Kaye,[3] the origin of the term *yāvat tāvat* is possibly connected with Diophantus's definition of the unknown quantity as "containing an indeterminate or undefined multitude of units" (*pléthos monádon áoriston*). Such a conjecture is too far fetched to be reliable It should be objected on other reasons also. For instance *yāvat tāvat* stands on a principle fundamentally different from that of Greek *pléthos monádon áoriston.* Diophantus calls the unknown quantity *arithmos*, meaning "number" and denotes it by a symbol which is an abbreviation of that word or of its inflected forms.[4] The Hindu *yāvat tāvat* is neither a definition of the unknown nor its name, but a symbol for the unknown which has no connexion whatsoever with its name or its definition Kaye has not explained why the Hindus, if they were at all influenced by the Greek science of algebra in the selection of a symbol for the unknown quantity, have deviated from the Greek principle of selecting it

The word *yāvat tāvat* is closely akin to *yadṛcchā* in form and more so in import.[5] I presume that the former has originated out of the

[1] The reference here is to the use of the letters of the alphabet to represent the unknown He states, "Or letters are to be employed , that is the literal characters *k*, etc , as names of the unknown, to prevent the confounding of them " (Colebrooke, *Hindu Algebra*, pp 228-9) This practice again is originally due to "the ancient teachers of science," but not to Bhāskara himself

[2] Colebrooke, *Hindu Algebra*, p. 139 , also compare p. 228 , "For which (the unknown quantities) *yāvat tāvat* and the several colours are to be put to represent the values."

[3] Kaye, *Indian Mathematics*, Calcutta, 1915, p. 25.

[4] Heath, *Greek Mathematics* II, p 456

[5] Compare expressions like "putting *yadrcchā* in the vacant place (*śūnya sthāna*)" of the Bakhshâlî work and 'putting *yāvat tāvat* for the unknown (*ajñāta*) of later algebras What is called *śūnya sthāna* in the Bakhshâlî work is denoted by *ajñāta* in later times.

latter. According to the celebrated Sanskrit lexicographer Amarasiṁha (c. 400 A.D), *yāvat tāvat* denotes "measure" or "quantity" (*māna*).[1] He had probably in mind the use of that term in Hindu algebra to denote "the measure of the unknown quantity" (*avyakta māna*).[2] In this way it appears that the origin of the symbol *yāvat tāvat* is connected with the rule of false position in algebra [3]

Plan of writing equations.

In the Bakhshâlî mathematics two sides of an equation are written down one after the other in the same line without any sign of equality being interposed. Thus the equations

$$\sqrt{x+5}=s, \qquad \sqrt{x-7}=t,$$

appear as [4]

$$\left[\begin{array}{cccc|cccccc} \circ & 5 & yu\ m\bar{u} & \circ & sa & \circ & 7 & + & m\bar{u} & \circ \\ 1 & 1 & & 1 & & 1 & 1 & & & 1 \end{array}\right].$$

The equation

$$x+2x+3\times 3x+12\times 4x=300$$

is stated as [5]

$$\left[\begin{array}{c|cc|cc|cc} \circ & 2 & 1 & 3 & 3 & 12 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] d\d{r}\acute{s}ya\ 300$$

Sometimes the unknown quantity is not indicated. Thus the equation

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{5}=65,$$

[1] *Amara-kosa*—"यावत्तावच्च साकल्येऽवधौ मानेऽवधारणे"

[2] Compare "यावत्तावत् कालको अव्यक्तानां कल्पिता मानसंज्ञा" (*Bījagaṇita* p 7).

[3] For a different theory about the origin of *yāvat tāvat* by Sarada Kanta Ganguly, see *Bull Cal. Math Soc* , XVIII (1927), pp 73 74

[4] Folio 59, recto.

[5] Folio 23, verso. See also folios 21, 23, 24, recto, etc.,

is represented as [1]

$$\begin{array}{|c|c|c|c|}
1 & 1 & 1 & \textit{dṛśya} \;\; 65 \\
2 & 3 & 4 & \qquad\;\; 1 \\
& & & \\
\hline
\end{array}$$

This latter plan is followed in the *arithmetical* treatises of Śrīdhara and Bhāskara According to the former [2]

$$\begin{array}{|c|c|c|l}
1 & 1 & 1 & \textit{dṛśya} \;\; 2 \\
2 & 6 & 12 & \\
& & & \\
\cline{1-3}
\end{array}$$

means $$x-\left(\frac{x}{2}+\frac{x}{6}+\frac{x}{12}\right)=2$$

Bhāskara does not use the lines [3] It will be noticed that in all the aforementioned works the absolute term is called *dṛśya*, meaning "visible," which is sometimes abbreviated into *dṛ* A distinction is sometimes made in its connotation in the different works The problems in connexion with which the above equations arise are of the same kind in all the works. But in the Bakhshâlî work, the term *dṛśya* refers to the "gives," while in the other works it generally refers to the "remains." [4] There is, however, one instance in the *Līlāvatī* in which the connotation of *dṛśya* is exactly same as in the Bakhshâlî work [5] This term is closely related to *rūpa*, meaning "appearance," which is the name for the absolute term in the Hindu algebra We find thus the true significance of the Hindu name for the absolute term in an algebraic equation. It represents the visible or known portion of the equation while its remaining part is practically unknown or invisible.

The above plan of writing equations differs much from the plan found in Hindu algebra in which (1) two sides are usually written

[1] Folio 70, recto and verso (*c*) See also folio 69, verso.

[2] *Triśatikā*, pp. 13 *et seq.*

[3] *Līlāvatī*, pp. 11 *et seq*

[4] The term *dṛśya* occurs also in the *Ganita-sāra saṁgraha* (iv. 4) in the sense of "remainder."

[5] *Līlāvatī*, p 11.

one below the other without any sign of equality and (2) the terms of similar denominations are written below one another, the terms of absent denominations from either sides being indicated by putting zero as its co-efficient.[1]

Certain Complex Series.

As already stated, the author of the Bakhshâlî mathematics is well acquainted with the rule for the summation of series in arithmetical progression Indeed he gives considerable importance to its treatment. There are instances of the geometrical progression in the work[2] There are further elementary cases of a certain class of complex series, the law of formation of which is quite clear If a_1, a_2, a_3 . denote the successive terms of any series, we find series of the type,[3]

(1) $a_1+2a_1+3a_1+4a_1+ \ldots +na_1,$

(2) $a_1+2a_1+3a_2+4a_3+\ldots+na_{n-1}$

(3) $a_1+2a_1+3(a_1+a_2)+4(a_1+a_2+a_3)+\ldots$

(4) $a_1+(2a_1\pm b)+\{3a_1\pm(b+d)\}+\{4a_1\pm(b+2d)\}+\ldots$

(5) $a_1+(2a_1+b)+\{3a_2+(b+d)\}+\{4a_3+(b+2d)\}+\ldots$

(6) $a_1+(2a_1+b)+\{3(a_1+a_2)\pm(b+d)\}+\{4(a_1+a_2+a_3)$
$\pm(b+2d)\}+\ldots$

(7) $a_1+(a_1r+da_1)+\{a_1r^2+d(a_1+a_1r)\}+\{a_1r^3$
$+d(a_1+a_1r+a_1r^2)\}+\ldots$

[1] *Hindu Contribution*

[2] Folio 51, verso

[3] The series of these types occur respectively on folio 22, verso, 23, recto, 23, recto and verso, 25, verso and 26, recto, 24, recto, 24, verso, and 25, recto, 51, recto and verso.

Evidently the series (4), (5), (6) are obtained respectively from the series (1), (2), (3) with the help of the subsidiary series in arithmetical progression,

$$b+(b+d)+(b+2d)+ \quad .$$

Similarly the series (7) is formed by combining the series in geometric progression

$$a_1+a_1r+a_1r^2+a_1r^3+\ldots$$

with another series formed out of its terms in the following way:

$$a_1+(a_1+a_2)+(a_1+a_2+a_3)+$$

The law of sequence underlying the above series is fully known to the author, as is shown by his explanatory notes. He says *tadā vargam tu kārayet* ("then construct the series")[1] The series is called *varga* and the sequence *krama* The sequence of the third type is aptly called *yutivargakrama*, that of the sixth type *yutaguṇita-yutakrama* or *yutaguṇitarṇakrama* according as the upper or lower sign in the terms are taken.

Rule of False Position.

It has been stated before that in the Bakhshâlî mathematics, problems leading to solution of algebraic equations are generally solved by a method which was known in the middle ages, amongst Arabic and European algebraists, by the name of the Rule of False Position. We find in the Bakhshâlî mathematics two types of equations which are solved by this method[2]

(1) In the first type, the equation required to be solved is

$$f(x)=p.$$

The mothod indicated for its solution is to assume any arbitrary value g for x; it will give

$$f(g)=p', \text{ say}$$

[1] Folio 23, recto.
[2] *Bakh Ms*, § 71

Then the true value of x will be $\frac{gp}{p'}$.

(2) In the second type, the equation given is

$$ax+b=p.$$

If g be a value of x such that

$$ag+b=p',$$

then the correct value will be

$$x=\frac{p-p'}{b}+g.$$

The rule of false position is found in the works of most of the Arabic algebraists beginning with Al-khowârizmî (*c.* 825 A. D.). From them it was learnt by the European scholars in the middle ages In India, it is expressly followed in the *Līlāvatī* of Bhāskara. This led Kaye to surmise that this rule "was introduced into northern India after the time of Śrīdhara (xith cent)"[1] But such a surmise, it will be presently shown, is wholly baseless

Known to Mahāvīra.

The rule of "false position" has been applied in certain cases in the *Ganīta-sāra-samgraha* For instance, for finding out an unknown quantity (*avyakta, ajñāta*) the sum of the various fractional parts of which is known, Mahāvīra says [2]

"The given sum, when divided by whatever happens to be the sum arrived at in accordance with the rule (mentioned) before by *putting down one in the place of the unknown* (element in the compound fractions), gives rise to the (required) unknown (element) in (the summing up of) compound fractions"

We have a few other instances of this kind in the work.[3] Further Mahâvîra has applied the method of supposition in solving

[1] *Bakh. Ms*, § 72.

[2] *Ganita-sāra-samgraha*, iii 107 Compare the original expression in this work *rūpaṁ nyasyāvyakte* with the passage *śūnye rupaṁ dattvā* and similar other passages in the Bakhshâlî work (folios 25, verso and 26, recto, compare also folios 22, 23)

[3] *Ibid*, iii 122, 125. 132, 135-7.

certain geometrical problems[1] Hence Kaye is not truly accurate when he says · "Mahāvīra (ix cent.), however, uses the method in rather a special way in connexion with a geometrical problem."[2] In fact Mahāvīra has made more extensive use of the method in connexion with certain algebraical as well as geometrical problems. Still it is quite true that he has not made as general use of the method as is found in the Bakhshâlî work, or even in the *Līlāvatī*[3] In any case Kaye's hypothesis of the foreign import of the rule of false position into India after the eleventh century must be abondoned. It should be further noted that Mahâvîra (*c* 850) is a contemporary of the earliest Arab algebraist to use that rule, namely, Al-khowârîzmî (*c.* 825) Hence it is quite certain that the Hindus have not taken the *regula falsi* from the Arab scholars, if they have done so at all from a foreign nation[4]

Known still earlier in India

It should be observed that the rule of false position was resorted to by the Arab and European algebraists at the early stage of development of their science when there were no symbols. It almost disappeared from amongst them, as it is bound to do, with the introduction of a system of notations[5] It will be nothing unreasonable to expect that such had been the case with that rule in India too,

[1] *Ibid*, vii 112½, 221½ This should be more accurately called the geometrical prototype of the *regula falsi* of algebra For further information on this point *vide* infra p 51 fn.

[2] *Bakh Ms*, § 72. This statement of Kaye followed by another of same kind, "It (the *regula falsi*) occurs in no Indian work until the time of Mahāvīra" (§ 134), will obviously contradict his previous statament, "Its occurrence in the *Līlāvatī* therefore seems to indicate that it was introduced into northern India after the time of Śrīdhara (xith century)" (§ 72) Thus it appears that Kaye is not sure of his own grounds

[3] Certain problems in the Bakhshâlî work, *Līlāvatī* (pp 10 *et seq*), *Triśatikā* (pp 13 *et seq*) and *Ganita sāra saṁgraha* (iv 5-32), which are of the same kind but differ only in details, have been solved in the first two works expressly by the *regula falsi*, but not so in the other two works, though in them the unknown quantity has been tacitly assumed to be one

[4] It should be noted in this connexion that while there are ample proofs in the writings of the early Arab scholars of their heavy indebtedness to Hindu Mathematics it is still to be proved that the Hindus took anything in return from them

[5] Smith, *History* II, p. 437.

if it was ever followed here. Now it is an well established fact that the Hindus reached Nesselmann's third and the last stage of development of the science of algebra long before all the other nations of the world [1] They invented a good system of notations by the beginning of the seventh century of the christian era It has been laid down by Brahmagupta (628 A D) that a thorough knowledge of algebraic symbols (*varṇa*) is an essential qualification for a good algebraist.[2] We find mention of a symbol for the unknown even in the *Āryabhaṭīya* of Āryabhata (499 A D)[3] So the method of false position must have disappeared from India before that time Or it is at least bound to have been relegated to a very inferior position from that time. This will account for the absence of the method from the works of Āryabhata and Brahmagupta as well as for its limited application in the *Gaṇita-sāra-saṃgraha* of Mahāvīra. There is now left no direct evidence from the Hindu source to show that that method was followed in India before the fifth century A. D. Unfortunately no Hindu treatises on arithmetic or on algebra which can be definitely referred to that period has survived and come down to this day [4] There is, however, external evidence. A mediaeval Arabic writer of note, possibly Rabbi Ben Ezra (*b*. 1095) refers the origin of the rule of false position to India.[5] And if our hypothesis about the origin of the term *yāvat tāvat* for the unknown in Hindu algebra be true, it is in all probability so, then there will remain very little to doubt that the rule was known in India much earlier.

Bhāskara's use accounted for

Application of the rule of false position by Bhāskara can be truly and more reasonably accounted for in a different way than as a result of contact with foreign nations or on any other hypothesis. It is not certainly without any significance that Bhāskara has

[1] *Hindu Contribution.*

[2] Brahmagupta says "By the pulverizer, cipher, negative and affirmative quantities, unknown quantity, elimination of the middle term, colours [or symbols] and factum, well understood, a man becomes a teacher among the learned and by the affected square" (Colebrooke, *Hindu Algebra*, p 325)

[3] *Hindu Contribution*

[4] Leaving of course the Bakhshâlî work which is under discussion,

[5] Smith, *History* II, p. 437, footnote 1.

applied that rule nowhere in his treatise on algebra where lies its proper place, but in his treatise on arithmetic, and there too in a limited way Still more significant is the fact that one problem which occurs in his *Bījagaṇita* as well as in his *Līlāvatī* has been solved in the latter treatise by the method of the false position while in the former by the ordinary algebraic device of solving linear equations [1] Similar differential methods of treatment are noticed to have been followed in case of certain other problems which occur in both the works [2] Bhāskara has indeed been forced by circumstances to do the same For as is well-known it is not at all permissible to use in arithmetic the symbols and notations which are freely permitted to be used in, in fact, whose use is essential for algebra. So a method which can be creditably followed in one place, may have to be shunned in another Now Bhāskara is found to have included in his arithmetical treatise certain topics which should properly belong to algebra, *e g*, *Iṣṭa-karma* ("rule of supposition" or "operation with an assumed number"), *Varga-karma* ("operation relative to squares"), *Guṇa-karma* ("operation with multiplicators"), *Kuṭṭaka* ("pulverizer"), etc [3] Bhāskara has the following excuse for so doing :[4]

"Algebra is similar to arithmetical rules, (but only) appears as if indeterminate (*guḍa*) It is not indeterminate to the intelligent : it is not certainly six-fold but many-fold Arithmetic is the rule of three and algebra is fine sagacity What is unknown to the highly intelligent ? So it is spoken for the dull intellect."

In fact the difference between algebra and arithmetic is according to him very thin and lies in the demonstration of the rules.

[1] *Līlāvatī*, p. 12 and *Bīja-gaṇita*, p. 48, Colebrooke, *Hindu Algebra*, pp. 24 and 192.

[2] Colebrooke, *Hindu Algebra*, pp 30 (§ 67) and 212 (§ 133), 31 (§ 68) and 211 (§ 132); 45 (§ 106) and 195 (§ 111), etc

[3] *Kuṭṭaka* has been included into arithmetic by Mahāvīra, Aryabhaṭa II and Bhāskara, but not by Brahmagupta According to the eminent commentator Gaṇeśa it has been included into arithmetic for the purpose of gratifying such as are not conversant with algebra And he has also pointed out that they are treated there without algebraic forms (Colebrooke, *Hindu Algebra*, p. 112 footnote)

[4] *Līlāvatī*, p 15 Similar observations have been repeated in the *Bījagaṇita* (p 49) and *Siddhānta-Śiromaṇi* (*Golādhyāya*, *Prasnādhyāya*, verses 3, 5) These show that Bhāskara attached much importance to this view Compare Colebrooke, *Hindu Algebra*, p. xix.

He says :[1]

"Mathematicians have declared algebra to be computation joined with demonstration · else there would be no difference between arithmetic and algebra."

The truth of this dictum will be clearly in evidence in the treatment of the *guṇa-karma* in the *Līlāvatī* and the *madhyamāharaṇa* in the *Bījagaṇita*. Both are practically treatment of the quadratic equations. But whereas we are given only the well-known formula for the solution of such equations in the former work, we have in the latter the method of deriving that formula and that too by different writers. Now all those subjects will have to be treated without the help of the algebraic artifices. The method in most cases is what may be called the "rule of supposition," or "operation with an assumed number." That is, starting with a number arbitrarily assumed (*iṣṭa-rāśi* or simply *iṣṭa*), Bhāskara shows how to obtain a solution of any given problem, which is sometimes its exact solution or in other cases a particularly limited one.[2] The general solution of the problems of the latter class cannot be obtained without the help of algebra. In dealing with the topic *iṣṭa-karma*, the rule of supposition leads to an exact solution (and this has not escaped the notice of Bhāskara).[3] And that is what has been called the *regula-falsi* in the west. It is noteworthy that this method did not attain much importance in Bhāskara's works as it once did in the middle ages in the west.

Special terminology.

The technical terms which are generally employed in the Bakhshâlî mathematics are mostly same as in other Hindu treatises on mathematics. But there are a few which are bound to distinguish it at once from the rest. For instance, the common Hindu term for the reduction of fractions to a common denominator is

[1] Colebrooke, *Hindu Algebra*, p 227; *Bījagaṇita*, p. 127.

उपपात्तयुतं वीजगणितं गणका जगुः ।
न चेदेवं विशेषोऽस्ति न पाटीवीजयोर्यतः ॥

Compare also : "since the arithmetic of known quantity (*vyakta*)...is founded on that of unknown quantity (*avyakta*)" (*Bījagaṇita*, p. 1).

[2] Compare Colebrooke, *Hindu Algebra*, pp. 45, 46.

[3] *Līlāvatī*, p. 11.

savarṇana, which means "making of the same class," but according to the Bakhshâlî works it should be *sadṛśī-karana* ("making similar") or *hara-sāmya-karana* ("making the denominators equal").[1] These two terms, though a little diffused, can be clearly recognised to be very closely related to, and indeed precursors of the other term The word *savarna* is found only once in the Bakhshâlî mathematics as forming a part of another compound word, *kalāsavarna*, which refers to the fraction in general or at least to a particular kind of it [2] This term reappears in the sense of general fraction in the *Ganita-sāra-samgraha*[3] of Mahāvīra and a nearly equal term in the *Triśatikā*[4] of Śrîdhara. Now the term *savarnana* is commonly adopted in Hindu mathematics from the time of Āryabhata (499 A D) So its absence from the Bakhshâlî mathematics will strongly suggest to refer this work to a period anterior to the fifth century of the Christian era Two other terms to lead one to such a presumption are *sthāpana* and more particularly *nyāsa-sthāpana* It has been already pointed out that in the later Hindu treatises on mathematics, the common technical term for the statement of a problem is *nyāsa*, while in the Bakhshâlî mathematics it is more frequently called *sthāpana* and occasionally *nyāsa* or *nyāsa-sthāpana* [5] Now the compound *nyāsa-sthāpana* is redundant, for both the constituents of it bear the same significance, so that either would have been quite sufficient for the object in view Its occurrence, as also that of *sthāpana* in the place of *nyāsa*, very likely implies that the Bakhshâlî work must be referred to a period of transition before the introduction of the modern term *nyāsa* Again the usual Hindu term for the series, from the fifth century A D , is *średhī*, meaning "series" but corresponding term in the Bakhshâlî mathematics is *varga* which means "group."[6] This term is also used to denote the square of a quantity The term

[1] We purposely say should be For these two terms do not occur in the Bakhshâlî mathematics in the form in which they are stated But they will very logically follow from the phrases used in this connexion, *e g*,, *sadṛśaṃ kriyate* (folio 1, verso), *hara sāmye krte* (folio 17, recto) and *sadrsa kr(te)* (folios 30, verso and 35, recto, 67 and 69, recto),

[2] *Bakh*, *Ms* , folio 85, verso

[3] *Ganita-sāra-samgraha*, iii 1

[4] *Triśtikā* pp 7, 12, In this work the fraction is more commonly called *bhinna*, which literally means "broken part," it is also called *kalāsavarnana*

[5] *Vide supra*, p 9 [6] *Vide supra*, p 31.

saṃkalita which once became so prominent in Hindu mathematics as to be adopted also by the Arabs,[1] occurs once in the Bakhshâlî mathematics [2] The term *krama* for the "sequence" is not found in other mathematical treatises As has been already pointed out the name *rupana-karana* is also unique for it. There are a few other minor technical terms, specially in the titles of sub-sections dealing with particular classes of problems For example, the sub-section in the Bakhshâlî work dealing with the mixure of golds of different varieties is called *suvarna-ksaya* ("loss of gold"), [3] in the *Līlāvatī*[4] it is called *suvarna-ganita* ("computations relating to gold"), in the *Ganita-sāra-samgraha*, *suvarna kuttīkāra* or *suvarna-ganita*,[5] and in the *Triśatikā*, *suvarna-varna-parijñāna* [6] The rules dealing with interest is called *hundikāsamānayana* sūtras,[7] while the corresponding terms in all other works are different [8] One method in the Bakhshâlî work is called *ekarāśistu kalanā-ganita-prakriyā*.[9] We do not find it elsewhere A few other peculiar technical terms are .[10] *partha* meaning "series" and probably connected with *pārthakya* and a derivative of *pṛtha* ("several") ; *dhānta* meaning "instalment" ; *pravṛthi* meaning " original amount."

Symbols and notations.

The Bakhshâlî mathematics is particularly characterised by the absence of any kind of algabraic symbols and notations. Though it shows a fair degree of progress in the science of algebra, there is not even a specific notation to represent the unknown quantity. This must have retarded to a great extent any further progress in the science We have already noted how the lack of an efficient

[1] *Hindu contribution.*

[2] Folio 4, verso.

[3] *Bakh. Ms* , folio 16, verso *Idānīm suvarnaksayaṃ vakṣyāma.*

[4] *Līlāvatī*, p 24

[5] vi 169, *suvarnaganitarupakuttīkāra*

[6] *Triśatikā*, p, 25

[7] Folio 67, recto.

[8] *Ganita-sāra-samgraha*, vi, 21 (*vrddhividhāna*); *Līlāvatī*, p 29 (*miśraka-vyavahāra*), *Triśatikā*, p 23 (*miśra vyavahāra, bhāvyaka-vyavahāra* and *eka-patrī-karana*).

[9] Folio 50, verso,

[10] *Ind. Ant*, xvii, p 278

symbolism has given rise to a certain amount of ambiguity in the representation of an algebraic equation and how it, often times, also has led to the adoption of the method of "false position" or "supposition" for the solution of the equation. The lack of algebraic symbols has left a further marked effect in the work. It has necessitated the preservation of every detail of the workings of the solution of algebraic problems keeping up their generality throughout so that the final statement of the results should clearly present the whole formula involved. "Indeed the numerical quantities in those problems are treated almost like algebraic symbols."[1] Hence in this way the Bakhshâlî mathematics differs greatly from the rest of Hindu mathematics which manifests a good system of algebraic symbols.

There are no special signs for the arithmetical operations in the Bakhshâlî work. Any particular operation intended is generally indicated by placing the abbreviation (initial syllable) of a Sanskrit word of that import after, occasionally before, the quantity affected. Thus the operation of addition is indicated by *yu* (an abbreviation of *yuta*, meaning "added"),[2] subtraction by + which is from *kṣa* (abbreviated from *kṣaya* "diminished",[3] multiplication by *gu* (abbreviation of *guṇa* or *guṇita*, meaning "multiplied")[4] and the division by *bhā*

[1] *Bakh Ms*, § 41, compare also § 38

[2] For example see folio 59, recto, where

$$\begin{array}{cc} o & 5 \\ 1 & 1 \end{array}\ yu \quad \text{means } x+5 \quad \text{and} \quad \begin{array}{c} 11 \\ 1 \end{array}\ yu\ \begin{array}{c} 5 \\ 1 \end{array} \quad \text{means } 11+5$$

[3] *Vide supra*, p 18

[4] For example, we have

$$\begin{array}{ccccccccc} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 10 & gu \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{array} \quad \text{meaning } 3\times3\times3\times3\times3\times3\times3\times10\ (=21870)$$

(Folio 47, recto)

and

$$\begin{array}{c|cc||cc||ccc||ccc||} 0 & 1 & 3 & 2 & 5 & 3 & gu & 7 & 4 & gu & 9 \\ 1 & 1 & 2 & 1 & 2+ & 1 & & 2+ & 1 & & 2 \end{array}$$

meaning $x(1+\frac{3}{2})+\{2x(1+\frac{3}{2})-\frac{5x}{2}\}+\{3x(1+\frac{3}{2})-\frac{7x}{2}\}+\{4x(1+\frac{3}{2})-\frac{9x}{2}\}$

(Folio 25, verso).

The beginning and end of this illustration are mutilated but the restoration is certain. Hence Hoernle is not correct in stating that "the operation of multiplication alone is not indicated by any special sign" (*Ind, Ant*, xvii, p. 36) The occurrence of an abbreviation for multiplication has not been noticed by Kaye (vide *Bakh. Ms.*, § 62). See also folio 63, verso.

(abbreviated from *bhāga* or *bhājita*, "divided").[1] Of these again, the more systematically employed abbreviation is that for the operation of subtraction and next comes that of division In case of other two operations, the indicatory words are often written in full, or occasionally nothing is indicated at all In the latter case, the particular operation intended to be carried on is to be understood from the context

The principle of choosing abbreviations of the words of respective imports as the signs of the first four fundamental arithmetical operations, as found in the Bakhshâlî work, is not met with in other Hindu treatises on mathematics, or indeed in any early mathematics The only symbol of an elementary arithmetical operation which the early Greeks possessed, *viz*, that of subtraction, is now known to be in no way connected with the Greek word of that import.[2] Such is also the case with the later Hindu negative sign But the principle reappears in the middle ages in Italy in the works of Pacioli (1494 A.D) and others[3] for plus and minus only, and in Spain in the works of al-Qalasâdî (1480) probably in a more general way[4] The symbols of other arithmetical operations such as powers and roots, as also of factum, absolute term in an algebraic equation and sometimes also unknowns, have been chosen by the later Hindu mathematicians clearly on that principle.[5]

[1] For instance, see folio 13, verso, where

$$\left\| \begin{matrix} 1 & 1 & 1 & 1\ bh\bar{a} \\ 1 & 1 & 1 & 1 \\ 2+ & 3 & 4+ & 5 \end{matrix} \right\| \begin{matrix} 36 \\ 1 \\ \end{matrix} \Bigg| \text{ means } \frac{36}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{5})}$$

and folio 42, recto in which

$$\left| \begin{matrix} 40\ bh\bar{a} \\ 1 \\ \end{matrix} \right\| \begin{matrix} 160 \\ 1 \\ \end{matrix} \left\| \begin{matrix} 13 \\ 1 \\ 2 \end{matrix} \right\| \text{ means } \frac{160}{40} \times 13\tfrac{1}{2}$$

[2] Heath, *Greek Mathematics* II, p 459 Smith on the other hand conjectures that it might have been so connected (Smith, *History* II, p 396)

[3] Smith, *History* II, p 397.

[4] F Wœpcke, "Recherches sur l'Histoire de sciences mathematiques chez les orientaux", *Journal Asiatique*, t iv (1854), pp 348 ff , *Cf* also his "Note sur des notations algébriques employéas par les Arabes", *C R*, t 39 (1854), p 162

[5] See Colebrooke, *Hindu Algebra*, pp. x—xiii.

In the Bakhshâlî work,[1] the square root of a quantity is indicated by writing after it *mū*, which is an abbreviation for *mūla*, meaning "root," while in the rest of the Hindu mathematics, it is indicated by *ka*, an abbreviation from *karaṇī*, meaning "surd."

So-called foreign influence.

Kaye thinks that the Bakhshâlî mathematics contains unmistakable signs of foreign, especially Muslim and Greek, influence.[2] Of the two main instances that he has cited in support of his contention, one refers to the rule for finding the approximate value of a surd quantity.[3] We have already shown how wrong are his notions about the existence and knowledge of that rule amongst the early Hindus. It was, indeed, known to them long before the Greeks and the Arabs. Therefore the occurrence of that rule in the Bakhshâlî mathematics is certainly no evidence of its having foreign influence. The other instance of Kaye is of doubtful value. It is as regards the use of the sexagesimal fraction. Kaye observes :

"Apparently there is only one purely arithmetical example of the use in the text and this example occurs, in connexion with a problem in arithmetical progression, on folio 6, verso, and 7, recto, where the fraction 178/29 is expressed as $6+8^{i}+16^{ii}+33^{iii}+6^{iv}\frac{6}{29}$ This sexagesimal fraction is actually written thus—

6
8
60
16 cha°
60
33 li°
60
6 vi°
60
śe° 6
29

[1] For instance, *vide* folio 59, recto

$$\begin{array}{|ccccc|}11 & yu & 5 & m\bar{u} & 4 \\ 1 & & 1 & & 1 \\ \hline\end{array} \quad \text{means} \quad \sqrt{11+5}=4,$$

and

$$\begin{array}{|ccccc|}11 & 7 & + & m\bar{u} & 2 \\ 1 & 1 & & & 1 \\ \hline\end{array} \quad \text{means} \quad \sqrt{11-7}=2,$$

compare also folio 67, verso. Kaye is wrong in thinking that *mū* indicates "squaring" (*Journ. Asiat. Soc. Beng.*, viii, 1912, p. 357)

[2] *Vide Bakh. Ms.*, §§ 119-120, 134. Compare also §§ 43, 44.

[3] *Vide* §§ 43, 69, 134. "There is not much doubt about the exegesis of this rule."

The upper three figures are missing in the manuscript but the restoration is certain.[1] Of the abbreviations *li°* stands for *liptā* (Gk *lepté*) which in Sanskrit works ordinarily means a minute of arc, or the sixtieth part of a degree, *vi°* stands for *viliptā*, ordinarily a second of arc while *śe°* stands for *śesham* or 'remainder'." [2]

Though certain weakness of this instance has been recognised by Kaye himself inasmuch as it contains apparent inaccuracies and obscurity—'the term *liptā* here applies to "third parts" instead of "first parts," and *viliptā* to "fourth parts"—and inasmuch as he fails to give the correct interpretation of the abbreviation *cha°* in it, he feels no hesitation in emphatically asserting that "No such example occurs in any early Hindu work and there is not the slightest doubt that it indicates direct western influence Indeed our author could have hardly provided us with a more conclusive piece of evidence."[3] We shall first of all point out that it is not a case of "the transformation of a simple fraction expressed in the ordinary way to the sexagesimal notation," as is supposed by Kaye. The fraction in question arises in course of the solution of the following example —

"A certain person goes 5 yojanas on the first day, and 3 yojanas more on each succeeding day. Another who travels 7 yojanas per day, has a start of 5 days When will they meet, say, O! the best of the mathematicians!"[4]

If x be the number of days in which the second man overtakes the first, then by the conditions of the problem, we shall have

$$7(5 + x) = \frac{x}{2}\left\{10 + (x - 1)\,3\right\},$$

$$\text{or} \quad 3x^2 - 7x - 70 = 0,$$

$$\text{whence} \quad x = \frac{7 + \sqrt{49 + 840}}{6} \text{ } dina \text{ (or days)},$$

[1] On reference to the manuscript it will be noticed that there is room for only one upper figure, but not three Hence from this consideration alone it may be suspected if the restoration is as certain as is assumed by Kaye.

[2] *Bakh Ms*, § 58

[3] *Ibid*, § 120

[4] Folio 6, recto,

the negative value of the radical is not considered in the work. Taking the approximate value of the surd, correct up to the second order, we get

$$x = \frac{178}{29}\ \textit{dina} = 6\frac{4}{29}\ \textit{dina}.$$

Expressing the fractional part of a *dina* in terms of the units of lower denomination, such as *ghaṭikā*, *vighaṭikā*, etc , we shall have

$$x = 6\ \textit{di.}\quad 8\ \textit{gha.}\quad 16\ \textit{vi}\quad 33^{iii}\quad 6^{iv}\ \frac{6}{29}$$

We do not name the units of the last denominations. In fact, such names are not found in any Hindu work But, as has been sufficiently indicated by the writer, each succeeding unit is one-sixtieth of the one preceding it

It will thus be noticed that what Kaye misrepresents to be a case of an abstract fraction is really a concrete case. What Kaye reads *cha°* is unmistakably *gha°*, abbreviation for *ghaṭikā* But even with this emendation, there remains much obscurity about the instance. Kaye's reading of *li°* is correct but we fail to see how to connect it with *ghaṭikā* The unit, *vighaṭikā* does not occur any where else in the Bakhshâlî work. Further the names of units have been misplaced in the manuscript. But this latter may be explained away as due to the fault of the scribe.

The above is not the only instance in India of the application of the approximate square root rule to a concrete case in which the result has been expressed in terms of the units of different denominations. For as early as the fifth century before the Christian era we find the instance,[1]

$$\sqrt{100{,}000{,}000{,}000}\ \textit{yojana}$$

$$= 316{,}227\ \textit{yojana}\quad 3\ \textit{gavyuti}\quad 128\ \textit{dhanu}\quad 13\tfrac{1}{2}\ \textit{angula}$$

and a little over

[1] *Jambudvīpaprajñapti*, Sūtra 3, *Jībābhigamasūtra*, Sūtra 82, etc.

The instance in question occurs in connexion with the calculation of the circumference of the Jambudvīpa which is of the shape of a circle and whose diameter is 100,000 *yojana*. The formula used in this calculation is

$$\text{circumference} = \sqrt{10 \times (\text{diameter})^2}$$

And this reappears in the later Jaina works also [1] Full details of the calculation of the above value and of similar other values are recorded in the notes of Siddhasenagani (c 56 B C) on the commentary of Umāsvāti (c 150 B C) on his own *Tattvārthādhigamasūtra* [2]

Other instances of the trace of foreign influence are stated by Kaye to be certain sorts of problems which lead to the solution of two particular types of linear equations. He does not, however, attach much importance to them for he apprehends that in those cases "it is possible that the problems reached the Bakhshâlî mathematics by way of other Indian works"[3] One set of those problems lead to the simple equations,[4]

$$c-\frac{1}{a_1}c-\frac{1}{a_2}(c-\frac{1}{a_1}c)- \quad =x, \qquad \ldots \quad (1)$$

or

$$x-\frac{1}{b_1}x-\frac{1}{b_2}(x-\frac{1}{b_1}x)-\ldots=x-T \qquad \ldots \quad (2)$$

Equations very similar to (2) appear in the mathematical papyrus of Akhmīm [5] There is, however, this difference that in the problems of the Bakhshâlî work, we are always given what is 'taken away' (T) from the original quantity (unknown) by the various specified operations, whereas in the problems of Akhmīm papyrus is given what is 'left' ($x—T$) after the operations Now the mathematical papyrus of Akhmīm is supposed to have been written between the 6th and 9th centuries And problems leading to equations similar to (1) and (2) are well known in the Hindu mathematical treatises written in that period, e g, *Triśatikā* (c. 750)[6] and *Ganita-sāra-saṁgraha* (850).[7] They are probably contemplated in a rule of *Brāhma-sphuṭa-siddhānta* (628) as is suggested by the illustrative example of the commentator Prithudakasvâmî (860).[8] Further there are reasons

[1] *Vide* for instance *Jambudvīpasamāsa* of Umāsvāti (c. 150 B. C.), ch. i; *Trailokyadīpikā*, and *Laghukṣetrasamāsa* of Ratnaśekharasūri (1440 A D,)

[2] *Tattvārthādhigamasūtra* with the commentary of Umāsvāti and notes of Siddhasenagani, Part I, edited by H R Kapadia, Bombay, 1926, pp 258 260.

[3] *Bakh Ms*, § 120 [4] *Ibid*, § 89.

[5] Heath, *Greek Mathematics* II, p 544 [6] p 11.

[7] iii 127-134, iv 29 32

[8] xii 9 and Prithudakasvâmî's commentary there on; *Cf.* Colebrooke, *Hindu Algebra*, p 283 fn

to believe that the Bakhshâlî work was written long before the period to which the composition of the mathematical papyrus of Akhmîm is referred. In such circumstances those problems cannot be called to show the stamp of foreign influence

The other set of examples give simultaneous linear equations of the type,[1]

$$x_1+x_2=a_1,\ x_2+x_3=a_2,\ \ ,\ x_n+x_1=a_n, \qquad . \quad (3)$$

$$\text{or}\quad \textstyle\sum x-x_1=c-d_1x_1,\ \sum x-x_2=c-d_2x_2,$$
$$\textstyle\sum x-x_n=c-d_nx_n \qquad . \quad (4)$$

Some particular cases of (3), namely, when $n=3$

$$x_1+x_2=a_1,\ x_2+x_3=a_2,\ x_3+x_1=a_3 \qquad . \quad (5)$$

are evidently expressible in the form[2]

$$\textstyle\sum x-x_1=c_1,\ \sum x-x_2=c_2,\ \sum x-x_3=c_3 \qquad . \quad (6)$$

One problem involving five unknown quantities gives a similar set of equations. Equations of the type (6) are supposed by some to be a modification of the type of equations considered by the Greek Thymaridas and which are solved by his well-known rule Epanthema [3] This resemblance leads Kaye to suspect an ultimate Greek influence in the origin of those problems.[4] One point appears to be in favour of Kaye's view namely that the simple equations of the type (5) occur in the *Arithmetica* of Diophantus [5] But it should be noted that the method of solution followed in the Bakhshâlî work is quite different from those of Thymaridas and Diophantus. Above all the equations (5) are but only particular cases of a more general type of simultaneous equations, namely (3), treated in the Bakhshâlî mathematics, the like of which are not found in Greek mathematics Equations of the type

[1] *Bakh Ms*, §§ 78, 79

[2] More general equations of this type connecting n unknown quantities occur in the *Āryabhaṭīya* (ii 29) of Āryabhaṭa (499 A D)

[3] This supposition has been disputed by Rodet (*Leçons de calcul d'Āryabhaṭa*) and Sarada Kanta Ganguly (*Journ. B O R Soc*, xii (1926), pp 88 *et seq*) The latter writer has ably shown that the so-called relation between the Hindu and Greek types of general simultaneous equations is based on misapprehension.

[4] *Bakh. Ms*, § 120

[5] I 16 *et seq*, Heath, *Greek Mathematics* II, p. 486.

(4) differ considerably in form, as also in the method of solution from the type of equations considered by Iamblichus who reduced them to a type to which Thymaridas's rule applies [1] Thus though there is a relation, that too in a modified way, in some particular cases, there is much more difference in other respects So we shall be entitled to reject the views of Kaye It will thus appear that Kaye has failed to establish his hypothesis of foreign influence in the Bakshhâlî mathematics In fact there may be very little, if any.

Impracticable problems

There are certain examples in the Bakhshâlî mathematics about which it may be rightly said that though there is nothing wrong in the mathematical principles which they are to illustrate, all the conditions of the problems cannot possibly be realised in life For instance, consider the following question [2]

> "Certain king gives away in succession one-half, one-third and one-fourth of his money, he gives 65 in total How much money he had in the beginning?"

It will be obtained easily that the king had originally 60 coins. Now it may be very rightly asked how is it possible for one to give out more than what he possesses? Another impracticable problem of this kind occurs on folio 69, recto.[3]

There is another set of examples which are specially notable inasmuch as their solution calls forth much mathematical skill and ingenuity of the commentator but which on the whole are highly improbable. Solution of each of those problems leads to the determination of the number of terms of a series in arithmetical progression whose first term, common difference and sum are given. And singularly enough, in every case of them, the number of terms comes out to be irrational, so that its exact value cannot be determined at all. Hence it is found that the problems do not admit of a real solution. One such problem we have already referred to on page 42. There the

[1] Heath, *Greek Mathematics* I, pp. 94 96

[2] Folio 70, recto and verso (c). Portions of the text are missing; but the question can be easily restored with the help of the statements

[3] In this case the text is destroyed beyond restoration But the impracticability of the problem will be recognised from the portion of the statement which is left.

expression for the time in which two persons will meet contains a surd quantity. So the two persons will never meet But such an answer does not seem to have been aimed at by the framer of the problem. His original mistake has been in the selection of elements which are incompatible. Other problems of this kind are mutilated beyond restoration in their original form But their true nature can be readily recognised from the portions of their statements which are still left [1] It should be pointed out that those problems probably deceived also the commentator to think sometimes that the number of terms of a series in arithmetical progression can be fractional.[2]

Relation with other Hindu treatises Brāhma-sphuṭa-siddhānta

Hitherto our object has been to treat mainly of those matters in respect of which the Bakhshâlî mathematics will be distinguished from the rest of Hindu mathematics. Such a study has, of course, its worth in the help that it renders in estimating the true character as well as the proper value of the Bakhshâlî mathematics. We shall now look up for those matters of resemblance which will suggest a possible connexion, more or less close, of the Bakhshâlî work with the one or the other of the remaining Hindu works on mathematics. Indeed without such an enquiry the present study will remain incomplete.

Hoernle thinks that the Bakhshâlî work bears a "peculiar connection" with the *Brāhma-sphuṭa-siddhānta* of Brahmagupta. He has pointed out that "there is a curious resemblance between the fiftieth *sūtra* of the Bakhshâlî arithmetic or rather with the algebraical example occurring in that *sūtra*, and forty-ninth (*sic*) *sūtra* of the chapter on algebra in the *Brahma-Siddhānta*."[3] The *sūtras* in question are in respect of the solution of the quadratic indeterminate equations of the type

$$\sqrt{x+a}=s^2,\ \sqrt{x-b}=t^2.$$

The *sūtra* in the Bakhshâlî work[4] is much mutilated, but can be

[1] *Bakh Ms*, § 85.

[2] For instance *vide* folio 7, recto, where the number of terms is stated to be (*tatra padam*) 178/29 On folio 45, recto, the *pada* is also big a fraction as 59425/49200 Compare also folio 46, recto

[3] *Ind Ant.*, xvii, p 37, compare also pp 40, 46, 47.

[4] *Bakh. Ms*, folio 59, recto.

partially restored from the solution. "The sum of the additive and subtractive numbers is divided by an assumed number; the quotient lessened by the same number and halved, is squared and added to the subtractive number" That is,

$$x=\left\{\frac{1}{2}\left(\frac{a+b}{m}-m\right)\right\}^2+b,$$

where m is any assumed number The solution given by Brahmagupta is exactly the same[2] There is also resemblance between the two works in the matter of solution of the other type of the quadratic indeterminate equations that is still preserved in the Bakhshâlî work, namely

$$xy-bx-cy-d=0.$$

The solution obtained is[3]

$$x=\frac{bc+d}{m}+c,\ y=b+m\ ;$$

where m is an assumed number. This is closely like the solution found in the *Brāhma-sphuṭa-siddhānta*,[4] but it differs considerably from the solutions given by Mahāvīra[5] and Bhāskara.[6] A still more noteworthy point is that Brahmagupta has admittedly taken the solution from an earlier work which is not known now.[7] Bhāskara does not treat of the other type of indeterminate equations noted above and Mahāvīra's solution of the same is very considerably different[8]

There are also other points of relation between the Bakhshâlî work and the *Brāhma-sphuṭa-siddhānta*. In Hindu mathematics fractions

[1] *Ind Ant*, xvii, p 44

[2] *Brāhma sphuta-siddhānta*, xviii 73, 84

[3] *Bakh Ms.*, folio 27, recto The text is very mutilated. Compare also § 82.

[4] xviii 60 Cf *Hindu Contribution*

[5] *Ganita sāra saṁgraha*, vi. 284 and vii 112½ *Vide infra*, p. 51 fn.

[6] *Bījaganita*, pp 123, Colebrooke, *Hindu Algebra*, p 270; *Hindu Contribution.*

[7] *Brāhma sphuta siddhānta*, xviii. 63; cf. Colebrooke, *Hindu Algebra*, p. 362, footnote 1

[8] Vide *infra*, p 5.

are usually divided into different classes (*jāti*). One class, which is truly of the most general class consisting of fractions of all the other varieties, is called in the Bakhshâlî work as *pañcamī jāti* ("the fifth class")[1] This is very significant For according to Śrīdhara,[2] Mahāvīra,[3] Skandasena and others,[4] there are six classes of fractions and the class referred to should be called, according to them, *Bhāga-mātā* (or "mother-fraction"). Bhāskāra has reduced the number of classes of fractions to four[5] It is only Brahmagupta who is known to recognise five classes of fractions.[6] Further we do not find in his work any kind of special technical names, as are commonly found in other Hindu treatises on mathematics. Hence, in the matter of classification of fractions the Bakhshâlî work is in complete agreement with the work of Brahmagupta There is an approximate formula in the *Brāhma-sphuta-siddhānta*,[7] which leads to

$$(a+x)^2=a^2+2ax,$$

where x is very small in comparison with a. This can be easily connected with the approximate square-root formula given in the Bakhshâlî work thus

$$\sqrt{a^2+2ax}=a+x$$

Putting ϵ for $2ax$, this will become

$$\sqrt{a^2+\epsilon}=a+\frac{\epsilon}{2a}$$

Ganita-sāra-samgraha

It has been observed by Kaye that "in some matters of detail the Bakhshâlî work more closely resembles the *Ganita-sāra-saṁgraha* of Mahāvīra than any other Indian work on mathematics."[8] This is true to a certain extent and indeed to a greater extent than what has been noticed by Kaye For those matters of detail, so far as they have been pointed out by him, consist of a few problems[9] and a very few names of measures[10] Those problems agree only to a

[1] *Bakh Ms*, folio 52, verso

[2] *Triśatikā*, pp. 10-12

[3] *Ganita-sāra-samgraha*, iii 54

[4] Referred to by Prithudakasvāmī (860 A D) Colebrooke, *Hindu Algebra*

[5] *Līlāvatī*, pp 6 7

[6] *Brāhma-sphuta-siddhānta*, xii 8, 9

[7] xii, 62

[8] *Bakh Ms*, § 119.

[9] *Ibid*, p 41, footnote 2, § 80; p. 44, footnote 1; p. 51, footnote 2.

[10] *Ibid*, pp 64, 67

partially restored from the solution : "The sum of the additive and subtractive numbers is divided by an assumed number ; the quotient lessened by the same number and halved, is squared and added to the subtractive number" That is,

$$x=\left\{\frac{1}{2}\left(\frac{a+b}{m}-m\right)\right\}^2+b,$$

where m is any assumed number The solution given by Brahmagupta is exactly the same.[2] There is also resemblance between the two works in the matter of solution of the other type of the quadratic indeterminate equations that is still preserved in the Bakhshâlî work, namely

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1 *Bakh Ms*, folio 52, verso

2 *Triśatikā*, pp. 10 12

3 *Ganita-sāra-samgraha*, iii 54

4 Referred to by Prithudakasvāmī (860 A D.). Colebrooke, *Hindu Algebra*

5 *Līlāvatī*, pp 6 7

6 *Brāhma-sphuta-siddhānta*, xii 8, 9

7 xii. 62

8 *Bakh Ms*, § 119

9 *Ibid*, p 41, footnote 2, § 80; p. 44, footnote 1, p. 51, footnote 2.

10 *Ibid*, pp. 64, 67

little extent in kind but differ greatly in other respects. And in the matter of measures, there are many times more points of disagreement between the two works than those of agreement.[1] There are, however, certain other matters which have not been noticed by Kaye, but which will appear to be strongly in support of his contention. Of these the two notable points are (1) the method of reducing fractions to the lowest common denominator and (2) the name *kalāsavarna* for fractions which occurs in those two works. There are a few motion problems of the same kind in the two works.[2] Another noticeable resemblance lies in the religious tenor underlying some of the problems in them.[3] But all these matters taken together are not sufficient, I think, to establish a direct, definite and near relation between the Bakhshālī work and the *Ganita-sāra-samgraha*. Indeed the matters of difference between them will heavily outweigh in importance the matters of resemblance. The problems, referred to above, are too simple to be of any particular interest from the mathematical point of view.[4] On the other hand, the two works differ to a very considerable extent

[1] Compare particularly the measures of time (p. 59), length (p. 61), money (p. 65) and weight (p 68) used in the two works

[2] *Ganita sāra samgraha*, vi 319 327½; *Bakh Ms*, folios 4 (recto and verso), 5 (recto), 7 (verso), 8 (recto) and 9 (recto) Compare also § 83.

[3] For instance there are mentions of offerings for the purpose of worship (*pūjā*) to the different Jīnas in the *Ganita-sāra-samgraha*, (pp. 10, 13, 22, 57, 62, 64, 72, etc) and to Śiva, Vasudeva and other gods and goddesses, as also of gifts to Brāhmaṇas, and others in the Bakhshâlî work (Folios 21-26, 33, 44, etc.; *cf.* § 52). Reference to such religious matters is rarely noticed in any other Hindu work on mathematics. Compare *Līlāvatī*, p. 11 (Colebrooke, *Hindu Algebra*, p. 24).

[4] The only problem of importance is the one leading to indeterminate equations of the type

$$x+y+z=d,$$

$$ax+by+cz=d.$$

The problems as stated in the two works differ in matters of detail. Again the one in the Bakhshâlî work is so mutilatad that it is almost impossible to say how far the methods of solution agree (*Bakh. Ms.*, § 80; *Gaṇita-sāra-samgraha*, vi 152 3). It is noteworthy that we miss in the *Gaṇita-sara-samgraha* anything of the kind of those motion problems of the Bakhshâlî work which are of special mathematical interest in view of the fact that they reqnire the application of the method of the approximate square root and of the consequent methods of reconciliation (*Bakh Ms*, §§ 85, 86).

in some matters involving important mathematical principles. For instance the indeterminate quadratic equations of the type

$$\sqrt{x+a}=s, \quad \sqrt{x-b}=t,$$

have been considered in both the works Mahāvīra's solution of the same is[1]

$$x=\frac{\{(a+b)(1+\alpha)/2\}^2-\alpha}{4}+1\pm\frac{a-b\mp\alpha}{2},$$

where α is the excess of $a+b$ over the nearest even number and where the upper or lower sign is to be taken according as $b>$ or $<a$.[2] This is obviously very cumbrous and limited. So it differs very materially from the solution given in the Bakhshâlî work which is more elegant as well as general The only other type of equations of the same class which occurs in this work, *viz*,

$$xy-bx-cy-d=0,$$

appears in the *Gaṇita-sāra-samgraha* in a specially limited way. And the principle underlying the method of solution given in it is altogether different.[3]

1 *Gaṇita-sāra samgraha*, vi. $275\frac{1}{2}$.

2 If the quantities a, b be fractional, it will be necessary, as has been pointed out by Rangacarya, to remove the fractional parts before the application of the formula. This can be easily done by multiplying both the equations by the square of the least common multiple of the denominators of the fractions The result obtained with these modified values of a, b, will have then to be divided by that square (*vide* Rangacarya's notes on vi, $278\frac{1}{2}$)

3 Mahāvīra has the following geometrical proposition for solution To construct a rectangle (or a square) whose area will be *numerically* (*saṁkhyayā*) equal to its perimeter, side, or diagonal, or a simple part of any one of them, or to an easy combination of two or more of them (vii $112\frac{1}{2}$) Expressed in terms of algebra, this proposition will lead to the solution of the indeterminate equation

$$xy=f(x, y),$$

where $f(x, y)$ is a simple function of known form Mahāvīra's solution of this proposition is as follows Take any other figure similar to the one required, then change its sides in the ratio of its corresponding element (*i e*, perimeter, etc) to its area. This will give the sides of the required figure. Divested of its geometrical garb this solution will stand thus Having obtained a general solution of the equation

$$x'^2+y'^2=z'^2,$$

The method of the "false position" is undoubtedly employed ir the two works. But whereas in the Bakhshâlî mathematics it figures as a very notable method of solving problems requiring the determina- tion of an unknown element, it has been relegated to a very inferio position in the mathematics of Mahāvīra leaving aside, of course, it geometrical prototype There are several examples in the two works which may be represented by[1]

$$x\left(\frac{1}{c_1}+\frac{1}{c_2}+.\ \ +\frac{1}{c_n}\right)=R=x-T,$$

$$c\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\ \ \left(1-\frac{1}{a_2}\right)=x,$$

or $$x\left(1-\frac{1}{b_1}\right)\left(1-\frac{1}{b_2}\right)\ .\left(1-\frac{1}{b_n}\right)=R'=x-T'.$$

calculate the functions $x'y'$ and $f(x', y')$ Then

$$x=x'\times\frac{f(x', y')}{x'y'}=f(x', y')/y',$$

$$y=y'\times\frac{f(x', y')}{x'y'}=f(x', y')/x'$$

The above method of solution is considered by Kaye to be a kind of the *regul falsi* (*Bakh Ms*, §§ 72, 134) It plays an important role, much more than wh has been supposed by Kaye, in the mathematics of Mahāvīra Indeed it has bee a powerful weapon at his hands in solving certain geometrical problems leading t indeterminate equations of the second degree (*Ganita sāra-samgraha*, vii. 122½, 221 cf *Hindu Contribution*)

Another problem of Mahāvīra which is more directly connected with an equa tion of this type is To find the increase or decrease of two given numbers (a, b so that the product of the resulting numbers will be equal to another given numbe (d) (vi 284) This will require the solution of the equation.

$$(a\pm x)(b\pm y)=d,$$

or $$xy\pm(bx+ay)+(ab-d)=0$$

This appears as general in form as the one occurring in the Bakhshâlî work. B the solution given by Mahāvīra is much cramped, so very considerably differen from that in the other works According to him

$$x=\frac{d\sim ab}{d+b},\ y=\frac{d\sim ab}{a+1}$$

Compare also vii 146

[1] *Bakh. Ms*, § 89, *Ganita-sāra samgraha*, iv 4

Similar examples also occur in other works, so there is nothing special about them [1] Only noticeable thing in them lies in some differences. For in the problems of the *Ganita-sāra-samgraha,* we always know R or R' (not T or T') which represents the quantity "remaining" and so is very appropriately called *drśya* ("visible" or "known"), whereas in the Bakhshâlî work, we know only T or T' which represents the quantity "taken away" and which is still called by the name *drśya* There are also other differences in matters of terminology, such as *średhī, varga, krama,* etc.

Other Hindu Works

There is no marked resemblance of the Bakhshâlî work with any other Hindu work on mathematics so as to suspect a possible relation It resembles the *Līlāvatī* of Bhāskara in the application of the method of false position for solution of certain algebraic equations (vide *supra* p. 36) The two works agree also in the manner of writing groups of fractions (vide *infra*) and in the similarity of a few examples. One problem in the Bakhshâlî work is proposed for solution to *sundarī,* "the beautiful one"[2] which will remind one of the similar mode of address in the *Līlāvatī* The cipher is found to have been employed in the two works in the place of an unknown quantity.

The resemblance between the Bakhshâlî work and the *Triśatikā* of Śrīdhara is still meagre It concerns about (1) the manner of writing fractions (*infra*), (2) method of writing equations, (3) the use of the term *rūpa* in connexion with an integer or the integral part of a mixed fraction

There are certain problems in the Bakhshâlî work which give equations of the type

$$\Sigma x - x_1 = a_1, \quad \Sigma x - x_2 = a_2, \quad \Sigma x - x_n = a_n$$

Equations of the same type are found in the *Âryabhatīya* and in no other Hindu works But Âryabhata's solution is different from that given in the Bakhshâlî work.

1 *Triśatikā*, pp. 13 *et seq*, *Līlāvatī*, pp. 11 *et seq*

2 Folio 34, recto.

Mode of writing fractions.

In the Bakhshâlî mathematics, the mode of writing fractions is the same as in the rest of Hindu mathematics For instance[1]

$$\begin{array}{c}15\\16\end{array} \text{ means } \frac{15}{16}, \quad \begin{array}{c}4\\1\\2\end{array} \text{ means } 4+\frac{1}{2}, \text{ and } \begin{array}{c}7\\1\\4+\end{array} \text{ means } 7-\frac{1}{4}.$$

Kaye wrongly asserts that this mode of writing fractions is "peculiarly" Arabic,[2] whereas the truth is on the contrary that the Arabs learned their mode from the Hindus.[3] The mode of writing groups of fractions is as follows

$$\left|\begin{array}{c|c|c|c}1&1&1&1\\4&3&6&12\end{array}\right| \text{ means } \frac{1}{4}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}$$

The expression

$$x\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)$$

is written as[4]

$$\left|\begin{array}{l}0\\1\\1\\2+\\1\\3\\1\\4+\\1\\5\end{array}\right| \quad \text{or} \quad \left|\begin{array}{ccccc}0&1&1&1&1\\1&1&1&1&1\\&2+&3&4+&5\end{array}\right|$$

and[5]

$$\begin{array}{c|l|l|}1&1&1\\2&4&5\\&1&1+\\&2+&4\\&&1+\\&&2\end{array}$$

[1] *Bakh Ms*, folio 12, verso.

[2] *Journ Asiat Soc Beng* III (1907), pp 502-3.

[3] *Hindu Contribution*, compare reference given therein

[4] Folio 13, recto and verso, *cf.* folios 10 15.

[5] Folio 52, verso

means

$$\frac{1}{2}+\frac{1}{4}\left(1-\frac{1}{2}\right)+\frac{1}{5}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{2}\right)$$

Exactly the same mode of writing groups of fractions have been followed in the works of Śrīdhara,[1] Pṛthudakasvāmī[2] and Bhāskara.[3]

Age of the Bakhshâlî Work.

One of the most difficult problems in the history of Hindu mathematics is that of the age of the Bakhshâlî work The scholarly attempt of the previous writers at fixing the age of the work led, we have seen before, to various estimates ranging from the early centuries of the Christian era to near about the twelfth century They largely based their estimates on the *literary* and *palaeographic* evidence. Our estimate of the age of the Bakhshâlî work will be based primarily and solely on the *historical* grounds In the absence of any other direct evidence, there can be no better guides in that respect than the mathematical principles, symbols and terminologies employed in the work. On *historical* arguments the age of the Bakhshâlî work must be placed nearer the time supposed by Hoernle than that by Kaye

As the present manuscript has been shown to reveal works of different strata as regards its character, it will be necessary to define clearly what we mean by the Bakhshâlî work. By it we always refer, not to the present manuscript, but to the work contained in it. The latter has been proved to be a perpetual commentary on an earlier treatise containing some mathematical *sūtras* (rule) together with a few illustrative examples The present manuscript has further been shown to be not the original of the commentary, but an imperfect copy of the same.

The *palaeographic* arguments can fix at best the age of the present manuscript But who knows how many years or centuries had elapsed since the date of the composition of the original Bakhshâlî sūtras till the time when a commentary on the same was written or

[1] *Triśatikā*, pp 11 *et seq*

[2] Colebrooke, *Hindu Algebra*, p 283 footnote, Compare also p. 15, footnote.

[3] *Līlāvatī*, pp. 6 *et seq.*

again since the date of composition of the Bakhshâlî commentary till the time when the present copy was made? Evidence based on the *language* will give some idea in this respect, but we shall leave its determination to the hands of the experts

Let us recall to mind the principal characteristics of the Bakhshâlî mathematics The most notable ones are the absence of a symbol for the unknown and the consequent adoption of the method of false position for the solution of algebraic equations. Indeed owing to an inefficient system of symbolism, the numerical quantities have oftentimes been treated almost like algebraic symbols. It is known that the Hindus had a symbol for the unknown as early as the fifth century of the Christian era It is now forgotten how long ago the Hindus abandoned the rule of false position, at least as an important instrument of solving algebraic equations. It must be before that time. So these facts strongly suggest that the Bakhshâlî work should be referred to an age before the fifth century A.D [1] Other remarkable things in the Bakhshâlî mathematics are the application of the approximate square-root formula, the calculation of errors of different orders and the process of reconciliation. The formula is not found expressly stated in its entirety in any Hindu treatise on mathematics from the time of Āryabhaṭa (499 A.D.) onwards. But it appears to have been well understood in India about the beginning of the Christian era and in the few centuries just preceding it. The other two methods were probably known about the same time. They do not occur in any later works. So the Bakhshâlî work was in all probability written about that time.

There are a few technical terms in the Bakhshâlî work, such as *sthāpana* for "statement," *varga* or *pāṛtha* for "series," *dhānta* for "instalment", *pravṛtti* for "the original amount" and *rūpaṇa karaṇa* which have totally disappeared, whereas there are a few others, *e g.*, *sadṛśī-karana* or *harasāmya-karana* and *nyāsa-sthāpana*, which can be undoubtedly recognised to be precursors of the corresponding terms in the later Hindu mathematics. Hence the work should be referred to a stage in the growth and development of Hindu mathematics before its terminology took the present form.

[1] It is held also by Kaye that the Bakhshâlî work was written before the introduction of an algebraic symbolism into Hindu mathematics (§§ 72 and 134). But he erred about the time of the latter.

From the consideration of all those points of much *historical* importance, I am inclined to conclude that the original Bakhshâlî work was composed in the early centuries of the Christian era. While there is nothing whatever in the work incompatible with it, there are, on the contrary, a few other facts also to point to this period (vide *infra*).

Origin of the Bakhshâlî Mathematics.

It has sometimes been suspected if the Bakhshâlî mathematics is at all of Hindu origin This suspicion has originally been created by Kaye [1] and has recently found place into an advanced work on the history of mathematics [2] Hence it is necessary that the whole position should be carefully reviewed and cleared of unjustifiable doubts and conjectures.

Hoernle holds that the Bakhshâlî mathematics is entirely of Hindu origin [3] But he has been severely criticised by Kaye for this view. And in a spirit of violent opposition he goes so far as to remark that "the implication that the work is wholly Hindu in origin has never been proved" [4] Such a demand for a proof of the Hindu origin of the Bakhshâlî mathematics seems apparently to be preposterous For the work has been found on Hindu soil and is written in a language of the Hindus It exhibits many characteristics of Hindu mathematics Hence justice and equity demand that the *prima facie* conclusion should be that the work is of Hindu origin.

[1] This suspicion was first expressed by Kaye in 1907 He then not only categorically denied the arguments of Hoernle in favour of the antiquity and the Hindu origin of the Bakhshâlî mathematics, but also asserted on the contrary that "every one of these points seems to me to emphasize the fact that this work is not of pure Indian origin clearer evidence for a non Indian origin could not be given" (*Journ Asiat Soc Beng*, *III* (1907), p 502, and also pp 502-3) This standpoint he gave up in 1912 in his first exclusive contribution on the Bakhshâlî work. He then made only a covert hint (*Journ Asiat Soc Beng*, *VIII*, 1912, p 356). Up to this time he seems not to have seen the original Bakhshâlî manuscript In his recent work, an edition of the Ms , he has restated his suspicion on more than one occasion (§§ 43, 44) without any attempt to substantiate it, as he should have done

[2] Smith, *History* I, p 161

[3] *Ind Ant*, *XVII*, p 36 and pp 37 8

[4] *Bakh Ms*, § 127

And that conclusion can be abandoned only when there forthcome satisfactory proofs on the contrary and in no case before that. Kaye has failed to produce any such evidence. On the other hand as a result of scrutiny of the contents of the Bakhshâlî work, he is convinced to observe in his latest work [1]

"But, of course, this evidence of western influence [2] does not mean that the work was not Indian It is, indeed, almost as Indian as any other mathematical work of the period It contains reference to Hindu mythology and to Hindu deities and the language is Indian of a sort the script is an off-shoot of the classical script of northern India, the form of presentation is Indian; and the material of most of the examples is Indian"

To these facts we should add, what are still more important, that the scope of topics discussed in the Bakhshâlî work and the methods of their treatment bear a very close relation to those that are generally found in other works of undoubted Hindu origin. Of the few signs of western influence noticed by Kaye, two principal ones have been shown to be misconceived and others can be, at the most adverse view, doubtful cases. And more than these, Kaye has failed to produce any point of resemblance of the Bakhshâlî work with a non-Indian work. Hence there remains practically nothing to question the Hindu origin of the Bakhshâlî mathematics. Moreover if we remember that the Bakhshâlî work was written in an age when the Arabic civilisation was yet to be born and that it exhibits no trace of the principal characteristics of the Greek mathematics, its Hindu origin is assured.

Noteworthy omissions.

Before concluding this study of the scope and character of the Bakhshâlî mathematics, reference should be made to another feature of it. No study of an early Hindu treatise on mathematics can be said to be complete without a notice of it. It is the omission of the treatment of (*i*) indeterminate equations of the first degree (*kuttaka*), (*ii*) the so-called Pellian Equation (*varga-prakṛti*) and (*iii*) shadow of a gnomon (*chāyā*). *Kuttaka* seems to be the most favourite subject of Hindu mathematicians. Attention of all them

[1] *Ibid*, § 121 This opinion is hardly consistent with his suspicion about the Hindu origin of the Bakhshâlî mathematics

[2] The reference here is to those instances which we have criticised on pages 41 *et seq.*

from Âryabhata (499 A.D.) onwards was directed to its treatment. And one of their greatest achievements in mathematics is the general solution of the indeterminate equations of the first degree, more than a thousand years before its rediscovery in Europe by Euler. The Hindu mathematicians were so enamoured of this subject that they oftentimes included its treatment in their treatises on arithmetic, although they knew that it really belongs to the domain of algebra and is actually retreated there. Another most notable feature of the classical Hindu treatises of mathematics is the treatment of the so-called Pellian Equation. A great part of the algebraic treatises of Brahmagupta and Bhāskara are devoted to this topic and in this matter they anticipated the labours of Lagrange by obtaining its most general solution The treatment of shadow problems are found to be included in all the known Hindu treatises of mathematics So the absence of any reference to any one of those subjects in the Bakhshâlî work is very much noticeable. And it is all the more so on account of the fact that there is evidence of considerable skill in the treatment of simultaneous linear equations and certain indeterminate equations of the second degree If these omissions have any significance on the determination of the time of the Bakashâlî mathematics, they strongly suggest to a period about the beginning of the Christian era But too much attention cannot be paid to these omissions, for they may be only apparent The entire sections dealing with them might have been destroyed, though the possibility of such a consequence is not great.

P S —

After the above has been set into types, I have discovered a remarkable passage in an early Jaina canonical work composed about 300 B C or still earlier, which is bound to be considered very important for the history of Hindu mathematics It will also corroborate some of the views expressed by the present writer in the foregoing pages It is stated in the passage referred to (*Sthānāngasūtra*, Sūtra 747) that the topics for discussion in the science of calculation (*samkhyāna*) are ten in numbers, viz, *parikarma* ("fundamental operations"), *vyavahāra* ("subjects of treatment"), *rajju* ("rope," meaning "geometry"), *rāśi* ("heaps," meaning "mensuration of solid bodies"), *kalāsavarna* ("fractions"), *yāvat tāvat* ("as many as," meaning "simple equations"), *varga* ("quadratic equations"), *ghana*

("cubic equations"), *varga-varga* ("biquadratic equations") and *vikalpa* ("permutations and combinations"). Owing to the deterioration of the culture of mathematics amongst the later Jaina scholars and other Hindus in general, the commentator Abhayadeva-sūri (1050 A D) has committed several errors in explaining the scope of the above topics, especially of those relating to algebra. Still he has rightly hinted that the term *yāvat tāvat* is equivalent to *yadrcchā* or *vāñchā*, meaning "an arbitrary quantity" (cf *kāmika* of the Bakhshâlî mathematics). This will corroborate the views expressed above about the close relation between these two terms and (p. 27*f*) and the knowledge of the rule of supposition much earlier in India (p 34). It appears further that in the centuries preceding the Christian era that rule was regarded so important that the section of the science of mathematics devoted to its treatment was named after it. The exclusive and prominent use of the rule in the Bakhshâlî work strongly leads us to conclude, as before, that the work must have been composed near about the same time. It may also be noted that as the term *yāvat tāvat* has entered Hindu mathematics before (at least by five centuries) the time of Diophantus (c 275 A D), the father of Greek algebra, those who have attempted to connect it with the work of this latter writer in the hope of showing the influence of Greek algebra on Hindu algebra, will have now to admit that the balance of evidence is just on the contrary so that Diophantus might have got inspiration from India.

For further and fuller discussion of the passage, the reader is referred to the author's forthcoming papers on (1) *The Jaina School of Mathematics* in the Bulletin of the Calcutta Mathematical Society, and (2) *Scope and Development of the Hindu Gaṇita* in the Indian Historical Quarterly.

2

ON THE CALCULATION OF THE ZEROS OF LEGENDRE POLYNOMIALS

BY

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1 The object of the present paper is to supply a method of obtaining approximate values for the zeros of Legendre polynomials of all orders. The principle involved in it hinges on the series (4) which expresses in a compact form the root ξ of the $(n+1)$th polynomial corresponding to a known root a of the nth polynomial The process of computing the roots consists in starting from the known zeros of a particular polynomial and building up zeros of successive polynomials by means of the simplified series (13), the values of the quantities n and a being properly modified It is believed that this method is new and suggestive of further research

In connection with the problem of mechanical quadrature Gauss* has given a table containing the zeros of these polynomials up to $n=7$

2. From the well known recurrence formula

$$(n+1)\ P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$$

it follows that the equation $P_{n+1}\ (x)=0$ is equivalent to

$$P_n(x)-\frac{n}{2n+1}\ \frac{P_{n-1}(x)}{x}=0 \qquad . \quad (1)$$

Let a denote a known root of $P_n(x)=0$ and ξ the corresponding root of $P_{n+1}\ (x)=0$ just *beyond* a †

Now write (1) in the form

$$\phi(x)=\phi(a)+\alpha f(x) \qquad .. \quad (2)$$

* Werke, III Bd Methodus nova integralium valores per approximationem inveniendi

† The zeros of $P_n\ (x)$ are interlaced with those of $P_{n+1}\ (x)$

where $\phi(x)=\mathrm{P}_n(x)$, $\phi(a)=0$,

$$a=\frac{n}{2n+1}\ ,\ f(x)=\frac{\mathrm{P}_{n-1}(x)}{x},$$

then in accordance with a formula established in a previous paper*

$\psi(\xi)$ is given by means of the series

$$\psi(a)+\sum_{r=1}^{\infty}\frac{a^r}{r!}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{r-1}\frac{\{f(a)\}^r\psi'(a)}{\phi'(a)} \tag{3}$$

In particular, ξ is given by the series

$$a+\sum_{r=1}^{\infty}\frac{a^r}{r!}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{r-1}\frac{\{f(a)\}^r}{\phi'(a)} \tag{4}$$

3 I proceed now to express the co-efficient of a^r in (3) in a more explicit form Let us start from

$$\frac{1}{r!}\left(\frac{1}{\phi'(x)}\frac{d}{dx}\right)^{r-1}\frac{\{f(x)\}^r\psi'(x)}{\phi'(x)} \quad \ldots \tag{5}$$

and afterwards put $x=a$

Using the recurrence formula

$$(x^2-1)\frac{d}{dx}\mathrm{P}_n(x)=n\{x\mathrm{P}_n(x)-\mathrm{P}_{n-1}(x)\}$$

$$i.e.,\ (x^2-1)\ \phi'(x)=nx\{\phi(x)-f(x)\}$$

$$\text{we get } f(x)=\phi(x)+\frac{1-x^2}{nx}\ \phi'(x),$$

$$\text{whence } \{f(x)\}^r=\sum_{m=0}^{r}\binom{r}{m}\phi^{r-m}\phi_1^{\,m} \tag{6}$$

where we write ϕ_1 for $\dfrac{1-x^2}{nx}\ \phi'(x)$ and ϕ for $\phi(x)$.

* This *Bulletin*, Vol. XIX, pp. 21-24.

Substituting the value of $\{f(\iota)\}^r$ in (5), we can express it in the form

$$\frac{1}{r!}\left(\frac{d}{d\phi}\right)^{r-1}\sum_{m=0}^{r}\binom{r}{m}\phi^{r-m}\ \mathrm{F}_m \tag{7}$$

where F_m stands for $\dfrac{\phi_1{}^m\psi'}{\phi'}$, *i.e.*, $\left(\dfrac{1-\iota^2}{nx}\right)^m\{\phi'(x)\}^{m-1}\psi'(x)$

Remembering that $\phi(a)=0$, it is easy to see that the value

of $\left(\dfrac{d}{d\phi}\right)^{r-1}\phi^{r-m}\mathrm{F}_m$ when $x=a$

is 0 if $m=0$, ... (*i*)

$(r-1)!\ \mathrm{F}_1(a)$ if $m=1$ ... (*ii*)

$\binom{r-1}{m-1}(r-m)!\left(\dfrac{1}{\phi'(a)}\dfrac{d}{da}\right)^{m-1}\mathrm{F}_m(a)$ if $m>1$ but $<r$ ... (*iii*)

Hence an alternative form for the co-efficient of a^r in (3)

$$\text{is}\sum_{m=1}^{r}\frac{1}{m!}\binom{r-1}{m-1}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-1}\mathrm{F}_m(a) \quad \dots \tag{8}$$

4 Let us now consider

$$\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-1}\mathrm{F}_m(a),$$

where $\mathrm{F}_m(a)=\dfrac{1}{n^m}\left(\dfrac{1}{a}-a\right)^m\{\phi'(a)\}^{m-1}\ \psi'(a)$.

As $\psi'(a)$ is at our choice a typical term in $\mathrm{F}_m(a)$ may be taken to be

$$a^\mu\left(\frac{1}{a}-a\right)^\nu\ \{\phi'(a)\}^{m-1}$$

where μ, ν are arbitrary constants

Denote $\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-1} a^{\mu}\left(\frac{1}{a}-a\right)^{\nu}\{\phi'(a)\}^{m-1}$ by $T_{m,\mu,\nu}$,

then as

$$\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-1} a^{\mu}\left(\frac{1}{a}-a\right)^{\nu}\{\phi'(a)\}^{m-1}$$

$$=\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-2}\left[\left\{\mu a^{\mu-1}\left(\frac{1}{a}-a\right)^{\nu}-\nu a^{\mu}\left(\frac{1}{a}-a\right)^{\nu-1}\right.\right.$$

$$\left.\left.\times\left(\frac{1}{a^2}+1\right)\right\}\{\phi'(a)\}^{m-2}+(m-1)a^{\mu}\left(\frac{1}{a}-a\right)^{\nu}\{\phi'(a)\}^{m-3}\phi''(a)\right]$$

we have

$$T_{m,\mu,\nu}=\mu T_{m-1,\mu-1,\nu}-\nu(T_{m-1,\mu-2,\nu-1}+T_{m-1,\mu,\nu-1})$$
$$+(m-1)\,S,$$

where $S=\left(\frac{1}{\phi'(x)}\frac{d}{dx}\right)^{m-2} x^{\mu}\left(\frac{1}{x}-x\right)^{\nu}\{\phi'(x)\}^{m-3}\phi''(x)$, $x=a$

Now $x\left(\frac{1}{x}-x\right)\phi''(x)=2x\,\phi'(x)-n(n+1)\phi(x)$,

therefore S consists of two parts one of which is

$2T_{m-1,\mu,\nu-1}$ and the other is

$$-n(n+1)\left(\frac{d}{d\phi}\right)^{m-2} x^{\mu-1}\left(\frac{1}{x}-x\right)^{\nu-1}\{\phi'(x)\}^{m-3}\phi(x), \text{ when } x=a$$

i e, $-n(n+1)(m-2)\,T_{m-2,\mu-1,\nu-1}$

Hence we have the formula

$$T_{m,\mu,\nu}=\mu T_{m-1,\mu-1,\nu}-\nu T_{m-1,\mu-2,\nu-1}+\{2(m-1)-\nu\}$$
$$\times T_{m-1,\mu,\nu-1}-n(n+1)(m-1)(m-2)T_{m-2,\mu-1,\nu-1} \qquad (9)$$

connecting consecutive T-functions

5. It is easily inferred that $T_{m,\mu,\nu}$ is expressible in the form

$$a^{\mu-2(m-1)}\left(\frac{1}{a}-a\right)^{\nu-(m-1)}\left\{p_0+p_1a^2+p_2a^4+\quad+p_sa^{2s}+\right.$$
$$\left.+p_{m-1}a^{2(m-1)}\right\}$$

where $p_0, p_1, p_2 \quad p_{m-1}$ are rational integral functions of μ, ν

Representing $T_{m-1,\mu,\nu}$ and $T_{m+1,\mu,\nu}$ respectively in the analogous forms

$$a^{\mu-2(m-2)}\left(\frac{1}{a}-a\right)^{\nu-(m-2)}\left\{q_0+q_1\,a^2+q_2\,a^4+\right.$$
$$\left.+q_{m-2}\,a^{2(m-2)}\right\}$$

and $a^{\mu-2m}\left(\frac{1}{a}-a\right)^{\nu-m}\{r_0+r_1\,a^2+r_2\,a^4+\quad+r_m\,a^{2m}\}$

it can be shewn by applying (9) that

$$r_s=\mu\{p_s(\mu-1,\nu)-p_{s-1}(\mu-1,\nu)\}-\nu\{p_s(\mu-2,\nu-1)+p_{s-1}(\mu,\nu-1)\}$$
$$+2mp_{s-1}(\mu,\nu-1)-n(n+1)m(m-1)\{q_{s-1}(\mu-1,\nu-1)$$
$$-q_{s-2}(\mu-1,\nu-1)\} \tag{10}$$

The formula (9) or (10) enables us to calculate the T-functions successively.

6 Let us now take the series (4) The co-efficient of a^r in this series for ξ is by (8)

$$\sum_{m=1}^{r}\frac{1}{n^m m!}\binom{r-1}{m-1}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{m-1}\left(\frac{1}{a}-a\right)^m\{\phi'(a)\}^{m-1}$$

$$\text{or } \sum_{m=1}^{r}\frac{1}{n^m m!}\binom{r-1}{m-1}T_{m,0,m} \quad \ldots \tag{11}$$

As the zeros of Legendre polynomials lie symmetrically in the interval $(-1, 1)$ it is convenient to use the formula for ξ^2 which may be written

$$a^2+\sum_{r=1}^{\infty} a^r \sum_{m=1}^{r} \frac{2}{n^m m\,!}\binom{r-1}{m-1} T_{m,1,m} \qquad (12)$$

Now $T_{1,1,1}=a\left(\frac{1}{a}-a\right)$,

$$T_{2,1,2}=-a^{-1}\left(\frac{1}{a}-a\right)(1+a^2),$$

$$T_{3,1,3}=a^{-3}\left(\frac{1}{a}-a\right)\{6-2(n^2+n+1)a^2+2n(n+1)a^4\},$$

$$T_{4,1,4}=-a^{-5}\left(\frac{1}{a}-a\right)\{60-12\,(2n^2+2n+5)\,a^2$$

$$+4(5n^2+5n+2)a^4+4n(n+1)a^6\}$$

and so on

The simplest formula for ξ^2 is ultimately

$$a^2+\frac{2(1-a^2)}{2n+1}+\frac{(1-a^2)\{(2n-1)a^2-1\}}{a^2(2n+1)^2}$$

$$+\frac{2(1-a^2)\{(4n^2-2n)a^4-(n^2+4n+1)a^2+3\}}{3a^4(2n+1)^3}$$

$$+\frac{(1-a^2)}{3a^6(2n+1)^4}\{(12n^3-4n^2-n)a^6-(6n^3+20n^2+11n+2)a^4$$

$$+(6n^2+24n+15)a^2-15\}$$

$$+\quad\cdot\qquad\cdot \qquad (13)^*$$

* When $|a|=0$ or *small* the series evidently fails Hence some of the smaller roots of $P_{n+1}(x)=0$ will remain undetermined These roots can, however, be obtained by having recourse to known elementary relations existing among the roots.

7. When n is large the series (12) or (13) admits of further simplification. This can be effected by means of the following two properties of the T-functions

$$(1)\quad \frac{T_{2l+1,\,\mu,\nu}}{n^{2l+1}} = \frac{(-1)^l (2l)!}{n}\; T_{1,\,\mu-l,\,\nu-l} + O\left(\frac{1}{n^2}\right)$$

$$(2)\quad \frac{T_{2l,\,\mu,\,\nu}}{n^{2l}} = O\left(\frac{1}{n^2}\right),$$

l being a positive integer.

The above results follow from (9).

Now neglecting terms of the order $\frac{1}{n^2}$, the co-efficient of a^r in (12) may be expressed as

$$\frac{2}{n} \sum_{s=0}^{v} \binom{r-1}{2s} (-1)^s \frac{T_{1,\,1-s,\,1+s}}{(2s+1)} \quad . \quad (14)$$

where $v=\frac{r-1}{2}$ if r is odd and $\frac{r-2}{2}$ if r is even.

Substituting for $T_{1,\,1-s,\,1+s}$ (14) may be written

$$\frac{2}{n} \sum_{s=0}^{v} \binom{r-1}{2s} (-1)^s \frac{(1-a^2)^{1+s}}{a^{2s}(2s+1)}$$ which readily becomes

$$\frac{2}{nr} \sin^2\theta \sum_{s=0}^{v} (-1)^s \binom{r}{2s+1} \tan^{2s}\theta$$

or $$\frac{\sin 2\theta \sin r\theta}{nr \cos^r\theta} \quad \ldots \quad (15)$$

if we put $\cos\theta$ for a.

Hence *when n is large* (13) reduces to

$$\cos^2\theta + \frac{\sin 2\theta}{n} \sum_{r=1}^{\infty} \frac{\sin r\theta}{r\, 2^r \cos^r\theta} \quad \ldots \quad (16)$$

where $\cos\theta$ is a root of $P_n(x)=0$.

It is to be observed that the formula (13) and (16) hold good for *unrestricted values of n*

My best thanks are due to Prof Ganesh Prasad for the great interest he has taken in the preparation of this paper

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3

On the Summability (C, 1) of Legendre Series of a function at a point where the function has a discontinuity of the second kind

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The question of the summability (C, 1) of the Legendre series corresponding to a given function was first discussed by A. Haar.* The problem was subsequently taken up by S. Chapman** and T. H. Gronwall † The latter established the theorem·

"The arithmetic mean of the first order of Laplace's series of an absolutely integrable function, on the whole sphere, converges as n tends to infinity, at each point of continuity to the functional value"

A shorter proof of the theorem was given by F. Lukács. †† A more elegant proof of the same theorem was given by L. Fejér *** Towards the end of his paper, Fejér gives the following sufficient condition·

"The absolutely integrable function $f(\theta, \phi)$ on the sphere has at the north pole the absolute mean value zero, if

$$\frac{1}{K_\epsilon}\iint_{K_\epsilon} |f(\theta, \phi)|\, do \longrightarrow 0 \text{ with } \epsilon \longrightarrow 0,$$

* *A Haar*—"Über die Legendresche Reihe" (*Rendiconti del Circ Mat. di Palermo*, tomo 32, 1911)

** *S Chapman* —"On the general theory of summability with applications to Fourier's and other series" (*Quarterly Journal of Pure and Applied Mathematics* Vol 43, 1912)

† *T H Gronwall* —" Uber die Laplacesche Reihe" (*Mathematische Annalen*, Vol 74, 1913).

†† *F Lukács*—"Über die Laplacesche Reihe" (*Mathematische Zeitschrift*, Bd. 44, 1922)

*** *L Fejér*—"Über die Summabilität der Laplacesche Reihe durch arithmetische Mittel" (*Mathematische Zeitschrift*, Bd. 24, 1926)

where the integration runs on the spherical cup, which is limited by the circle of spherical radius $\theta = \epsilon$ (and contains the North pole and where $K_\epsilon = 4\pi \sin^2 \frac{\epsilon}{2}$ denotes the surface content of this spherical cup)."

It is now found that the theorem is not as general as was supposed by Fejér. Indeed, it does not hold for every function having a discontinuity of the second kind Although the illustrative example that he has taken has a discontinuity of the second kind, the function considered becomes zero at both the limits after integration with respect to ϕ.

In the present paper, two examples are given, in which the functions have discontinuities of the second kind at the point considered, but they do not vanish at the limits after integration with respect to ϕ. It has been shown that in these two cases, Fejér's sufficient conditions are not satisfied, although the corresponding Legendre series are summable (C, 1). For facilitating the proof, a number of lemmas have been established in Art. 1. In Art. 2, the general problem has been formulated, in Art. 3, the first example is treated in which the function is bounded and integrable. In Art. 4, a similar function has been taken, having an infinite discontinuity of the second kind at the origin, although the function is absolutely integrable.

My best thanks are due to Professor G. Prasad for encouragement and interest

1 Lemma 1 —*If*

$$\Phi(\theta) = \int_0^{1-\cos\theta} \frac{1}{t^m} \cos\frac{1}{t^n}\, dt,$$

then

$$\left|\frac{\Phi(\theta)}{(1-\cos\theta)^{n-m+1}}\right| \leq \frac{2}{n},$$

for all values of θ, *including* $\theta = 0$, *if* $n + 1 \geq m$,

Integrating by parts, we have

$$\Phi(\theta) = -\frac{1}{n}(1-\cos\theta)^{n-m+1}\sin\frac{1}{(1-\cos\theta)^n}$$

$$+\frac{n-m+1}{n}\int_0^{1-\cos\theta} t^{n-m}\sin\frac{1}{t^n}dt$$

i.e $$\left|\frac{\Phi(\theta)}{1-\cos\theta}\right| \leq \frac{1}{n}(1-\cos\theta)^{n-m}\left\{1+\left|\sin\frac{1}{(1-\cos\theta)^n}\right|\right\}$$

$$\leq \frac{2(1-\cos\theta)^{n-m}}{n},$$

so that $\frac{\Phi(\theta)}{1-\cos\theta}$ is a continuous function of θ, being equal to zero, when $\theta = 0$, if $n > m$

Moreover,

$$\left|\frac{\Phi(\theta)}{(1-\cos\theta)^{n-m+1}}\right| \leq \frac{2}{n},$$

for all values of θ, including $\theta=0$

Cor. 1 If $m=0$, $n=1$,

$$\Phi(\theta)=\int_0^{1-\cos\theta} \cos\frac{1}{t}\,dt$$

and $$\left|\frac{\Phi(\theta)}{1-\cos\theta}\right| \leq 2(1-\cos\theta)$$

Further,

$$\left|\frac{\Phi(\theta)}{(1-\cos\theta)^2}\right| \leq 2,$$

for all values of θ, including $\theta=0$.

Cor 2. If $n-m=1$,

$$\left|\frac{\Phi(\theta)}{(1-\cos\theta)^2}\right| \leq \frac{2}{n},$$

for all values of θ, including $\theta=0$

If $s_n^{(1)}(\cos\theta)$ denote the first arithmetic mean of the series $\sum_{m=0}^{\infty}(2m+1)P_m(\cos\theta)$ and

$$s_n^{(1)}(\cos\theta)=\frac{S_n^{(1)}(\cos\theta)}{n+1}$$

then

$S_n^{(1)}(\cos\theta)$

$$=P_0(\cos\theta)\frac{\sin^2(n+1)\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+P_1(\cos\theta)\frac{\sin^2 n\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+$$

$$+P_r(\cos\theta)\frac{\sin^2(n-r+1)\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\quad +P_n(\cos\theta)\frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}.$$

This is *Fejér's* result.*

LEMMA 2.—*If* $\left|\frac{\overline{F}(\theta)}{1-\cos\theta}\right| \leq A(1-\cos\theta)$, *for all values of* θ, *including* $\theta=0$, A *being a finite constant,*

$$\lim_{n=\infty}\frac{1}{n+1}\left|F(\theta)\,S_n^{(1)}(\cos\theta)\right| < A\,\theta^2,$$

if θ *is sufficiently small.*

We have,

$$\left|F(\theta).\,S_n^{(1)}(\cos\theta)\right|$$

* *Loc. cit* p 273.

$$\leq 2 \left| \frac{F(\theta)}{1-\cos\theta} \right| \left| P_0(\cos\theta)\sin^2(n+1)\frac{\theta}{2} + P_1(\cos\theta)\sin^2 n\frac{\theta}{2} + \ldots + P_n(\cos\theta)\sin^2\frac{\theta}{2} \right|$$

$$< 2 \left| \frac{F(\theta)}{1-\cos\theta} \right| \left\{ |P_0(\cos\theta)| + |P_1(\cos\theta)| + \ldots + |P_n(\cos\theta)| \right\}$$

$$< 2(n+1) \left| \frac{F(\theta)}{1-\cos\theta} \right|,$$

since $|P_n(\cos\theta)| \leq 1$, for all values of n and θ

Therefore,

$$\lim_{n=\infty} \frac{1}{n+1} \left| F(\theta)\, S_n^{(1)}(\cos\theta) \right| < \lim_{n=\infty} \frac{2(n+1)}{n+1} \left| \frac{F(\theta)}{1-\cos\theta} \right|$$

$$< 2A(1-\cos\theta) < 4A\sin^2\frac{\theta}{2} < A\theta^2,$$

if θ be sufficiently small, and this limit is zero when $\theta=0$

LEMMA 3 —*If* $\left| \dfrac{F(\theta)}{(1-\cos\theta)^2} \right| \leq A$, *for all values of* θ, *including* $\theta=0$, A *being a finite constant,*

$$\lim_{n=\infty} \frac{1}{n+1} \left| F(\theta)\cot\frac{\theta}{2}\, S_n^{(1)}(\cos\theta) \right| < 2A\theta,$$

if θ *is sufficiently small.*

We have

$$\left| F(\theta)\cot\frac{\theta}{2}\, S_n^{(1)}(\cos\theta) \right|$$

$$\leq 2 \left| \frac{F(\theta)}{1-\cos\theta}\cot\frac{\theta}{2} \right| \left| P_0(\cos\theta)\sin^2(n+1)\frac{\theta}{2} + P_1(\cos\theta)\sin^2 n\frac{\theta}{2} + \ldots + P_n(\cos\theta)\sin^2\frac{\theta}{2} \right|$$

$$< 2\left|\frac{F(\theta)}{(1-\cos\theta)^2}\right| |\sin\theta| \Big\{ |P_0(\cos\theta)| + |P_1(\cos\theta)| + \\ + |P_n(\cos\theta| \Big\}$$

$$< 2\,A\,|\sin\theta|\ (n+1)$$

$< 2\,(n+1)\,A\,\theta$, if θ be sufficiently small

Hence

$$\lim_{n=\infty} \frac{1}{n+1}\left| F(\theta)\cot\frac{\theta}{2}\ S_n^{(1)}\ (\cos\theta)\right| \leq 2\,A\,\theta$$

Lemma 4.—*If* $\left|\dfrac{F(\theta)}{(1-\cos\theta)^2}\right| \leq A$, *for all values of* θ, *including* $\theta=0$, A *being a finite constant,*

$$\lim_{n=\infty} \frac{1}{n+1}\left|\frac{F(\theta)}{1-\cos\theta}\sin\theta\Big\{\frac{dP_1}{dx}\sin^2\frac{n\theta}{2}+\frac{dP_2}{dx}\sin^2(n-1)\frac{\theta}{2}+ \\ \cdot + \frac{dP_n}{dx}\sin^2\frac{\theta}{2}\Big\}\right|$$

$< C'\theta$, *if* θ *is sufficiently small,* C' *being another constant*

We know that

$$\frac{dP_n}{dx}+\frac{dP_{n+1}}{dx}=\frac{n+1}{1-x}(P_n-P_{n+1})$$

Therefore,

$$2\left(\frac{dP_1}{dx}+\frac{dP_2}{dx}+\ldots+\frac{dP_{n-1}}{dx}+\frac{1}{2}\frac{dP_n}{dx}\right)$$

$$=\frac{1}{1-x}\Big\{(P_0-P_1)+2(P_1-P_2)+3(P_2-P_3)+\ ..\ +n(P_{n-1}-P_n)\Big\}$$

$$=\frac{1}{1-x}\Big\{P_0+P_1+P_2+\ ..+P_{n-1}-nP_n\Big\}.$$

Further,

$$\frac{dP_n}{dx}-\frac{nP_n}{1-x}=-\frac{n}{1-x}\left\{\frac{xP_n-P_{n-1}}{1+x}+P_n\right\}$$

$$=-\frac{n}{1-x^2}\left\{P_n-P_{n-1}+2xP_n\right\}.$$

Therefore

$$\tfrac{1}{2}\left|\frac{dP_n}{dx}-\frac{nP_n}{1-x}\right|\leq\frac{n}{2(1-x^2)}\left\{|P_n|+|P_{n-1}|+2|x|\,|P_n|\right\}$$

$$\leq\frac{2n}{1-x^2}$$

Hence,

$$\left|\frac{dP_1}{dx}+\frac{dP_2}{dx}+\quad+\frac{dP_n}{dx}\right|$$

$$\leq\frac{1}{2(1-x)}\left\{|P_0|+|P_1|+..+|P_{n-1}|\right\}+\tfrac{1}{2}\left|\frac{dP_n}{dx}-\frac{nP_n}{1-x}\right|$$

$$\leq\frac{n}{2(1-x)}+\frac{2n}{1-x^2}\leq\frac{5n}{2(1-x)}\text{ if }x\geqq 0.$$

Since, by Hardy's form of Abel's Lemma,

$$\left|\frac{dP_1}{dx}\sin^2 n\frac{\theta}{2}+\frac{dP_2}{dx}\sin^2(n-1)\frac{\theta}{2}+\quad.+\frac{dP_n}{dx}\sin^2\frac{\theta}{2}\right|$$

$$\leq C\left|\frac{dP_1}{dx}+\frac{dP_2}{dx}+\quad+\frac{dP_n}{dx}\right|,\text{ C being a constant}$$

hence,

$$\left|\frac{F(\theta)}{1-\cos\theta}\sin\theta\left\{\frac{dP_1}{dx}\sin^2 n\frac{\theta}{2}+\frac{dP_2}{dx}\sin^2(n-1)\frac{\theta}{2}+.\right.\right.$$

$$\left.\left.+\frac{dP_n}{dx}\sin^2\frac{\theta}{2}\right\}\right|$$

$$\leq C \left| \frac{F(\theta)}{1-\cos\theta} \right| \, |\sin\theta| \, \left| \left\{ \frac{dP_1}{dx} + \frac{dP_2}{dx} + \quad + \frac{dP_n}{dx} \right\} \right|$$

$$< C \left| \frac{F(\theta)}{1-\cos\theta} \right| \, |\sin\theta| \, \left| \frac{5n}{2\,(1-\cos\theta)} \right|$$

$$< C \left| \frac{F(\theta)}{(1-\cos\theta)^2} \right| \, |\sin\theta| \, \frac{5n}{2}$$

$<\frac{5}{2}C\ An\theta$, if θ is sufficiently small.

Therefore

$$\lim_{n=\infty} \frac{1}{n+1} \left| \frac{F(\theta)}{1-\cos\theta}\sin\theta \left\{ \frac{dP_1}{dx}\sin^2 n\frac{\theta}{2} + \frac{dP_2}{dx}\sin^2(n-1)\frac{\theta}{2} + \right.\right.$$

$$\left.\left. + \frac{dP_n}{dx}\sin^2\frac{\theta}{2} \right\} \right|$$

$< \theta\, C'$, if θ is sufficiently small

LEMMA 5 —

$$(m+1)\sin(m+1)\theta\, P_0 + m\sin m\theta\, P_1 + \quad + \sin\theta\, P_m$$

$$= \frac{(m+1)(m+2)}{3\sin\theta} \left\{ P_m(\cos\theta) - P_{m+2}(\cos\theta) \right\}.$$

We know that

$$\frac{1}{(1-2rx+r^2)^{\frac{1}{2}}} = P_0(x) + rP_1(x) + r^2P_2(x) + \,.\,.+ r^nP_n(x) + .$$

$$= \sum_{m=0}^{\infty} r^m P_m(x) \qquad . \qquad .. \qquad (1)$$

and

$$\frac{1}{(1-2rx+r^2)^{\frac{3}{2}}} = \sum_{m=0}^{\infty} r^m \frac{dP_{m+1}}{dx} \qquad .. \qquad ... \qquad . \qquad (2)$$

Also

$$\frac{I}{(1-2rx+r^2)^{\frac{5}{2}}} = \frac{1}{3} \sum_{m=0}^{\infty} r^m \frac{d^2P_{m+2}}{dx^2} \qquad ... \qquad . \qquad (3)$$

Further, we have,

$$\frac{e^{\iota\theta}}{(1-re^{\iota\theta})^2}=\sum_{m=0}^{\infty}(m+1)r^m e^{\iota(m+1)\theta} \quad . \quad . \quad (4)$$

Multiplying (1) and (4) and arranging the terms as Cauchy-product, we have,

$$\frac{1}{(1-2rx+r^2)^{\frac{3}{2}}}\ \frac{e^{\iota\theta}}{(1-re^{\iota\theta})^2}=\sum_{m=0}^{\infty}r^m U_m, \quad . \quad (5)$$

where

$$U_m=(m+1)e^{\iota(m+1)\theta}P_0(x)+me^{\iota m\theta}P_1(x)+\ \ .+e^{\iota\theta}\ P_m(x)$$

But the left-hand side of (5)

$$=\frac{e^{\iota\theta}(1-re^{-\iota\theta})^2}{(1-2rx+r^2)^{\frac{5}{2}}}=\frac{e^{\iota\theta}-2r+r^2e^{-\iota\theta}}{(1-2rx+r^2)^{\frac{5}{2}}},\ \text{(where } x=\cos\theta)$$

$$=\frac{e^{\iota\theta}-2r+r^2e^{-\iota\theta}}{3}\sum_{m=0}^{\infty}r^m\frac{d^2P_{m+2}}{dx^2}\ \text{from (3)}$$

Hence equation (5) becomes

$$\frac{e^{\iota\theta}-2r+r^2e^{-\iota\theta}}{3}\sum_{m=0}^{\infty}r^m\frac{d^2P_{m+2}}{dx^2}=\sum_{m=0}^{\infty}r^m U_m \quad .. \quad ... \quad (6)$$

Equating the coefficients, of r^m on both sides, we get,

$$e^{\iota\theta}\ \frac{d^2P_{m+2}}{dx^2}-2\frac{d^2P_{m+1}}{dx^2}+e^{-\iota\theta}\ \frac{d^2P_m}{dx_2}=3U_n \quad .. \quad ... \quad (7)$$

Now, equating the imaginary parts on both sides, we have

$$\frac{\sin\theta}{3}\left(\frac{d^2P_{m+2}}{dx^2}-\frac{d^2P_m}{dx^2}\right)$$

$$=(m+1)\sin(m+1)\theta\ P_0+m\sin m\theta\ P_1+\ldots+\sin\theta\ P_m \quad .. \quad (8)$$

But, from the differential equations satisfied by P_m and P_{m+2}, we get,

$$(1-x^2)\left\{\frac{d^2P_{m+2}}{dx^2}-\frac{d^2P_m}{dx^2}\right\}$$

$$=2x\left\{\frac{dP_{m+2}}{dx}-\frac{dP_m}{dx}\right\}-(m+2)(m+3)P_{m+2}+m(m+1)P_m$$

$$=2x(2m+3)P_{m+1}-(m+2)(m+3)P_{m+2}+m(m+1)P_m$$

(by Christoffel's formula)

$$=2(m+2)P_{m+2}+2(m+1)P_m-(m+2)(m+3)P_{m+2}+m(m+1)P_m$$

$$=(m+1)(m+2)(P_m-P_{m+2})$$

Hence

$$\frac{\sin\theta}{3}\left(\frac{d^2P_{m+2}}{dx^2}-\frac{d^2P_m}{dx^2}\right)=\frac{(m+1)\,(m+2)}{3\sin\theta}(P_m-P_{m+2})$$

Therefore, equation (8) becomes,

$$\frac{(m+1)\,(m+2)}{3\sin\theta}\,(P_m-P_{m+2})$$

$$=(m+1)\sin(m+1)\theta\, P_0+m\sin m\theta\, P_1+\quad+\sin\theta\, P_m$$

Lemma 6—*If* $\dfrac{F(\theta)}{(1-\cos\theta)^2}$ *be a summable function in the interval* $(\cos\epsilon, 1)$, ϵ *being arbitrarily small, but fixed, then*

$$\lim_{n=\infty}\frac{1}{n+1}\int_0^{\epsilon}\frac{F(\theta)}{1-\cos\theta}\Big\{(n+1)\sin(n+1)\theta\, P_0+n\sin n\theta\, P_1$$

$$+\ldots+\sin\theta\, P_n\Big\}\,d\theta<\epsilon$$

We have, by Lemma 5,

$$(n+1)\sin(n+1)\theta\, P_0+n\sin n\theta\, P_1+\quad.+\sin\theta\, P_n$$

$$=\frac{(n+1)(n+2)}{3\sin\theta}\,\{P_n(\cos\theta)-P_{n+2}(\cos\theta)\},$$

hence, if $F(\theta)$ $F_1(\cos\theta)$,

$$\lim_{n=\infty}\frac{1}{n+1}\int_0^{\epsilon}\frac{F(\theta)}{1-\cos\theta}\{(n+1)\sin(n+1)\theta\,P_0+n\sin n\theta\,P_1+\ldots+\sin\theta\,P_n\}\,d\theta$$

$$=\lim_{n=\infty}\frac{n+2}{3}\int_0^{\epsilon}\frac{F(\theta)}{1-\cos\theta}\;\frac{P_n(\cos\theta)-P_{n+2}(\cos\theta)}{\sin\theta}\,d\theta$$

$$=\lim_{n=\infty}\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{1}{1+x}\{P_n(x)-P_{n+2}(x)\}\,dx \tag{9}$$

Now,

$$(n)\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{dx}{1+x}P_n(x)=0(\sqrt{n})$$

and

$$(n+2)\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{dx}{1+x}P_{n+2}(x)=0(\sqrt{n}),$$

therefore,

$$\lim_{n=\infty}\frac{n+2}{3}\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{dx}{1+x}\{P_n(x)-P_{n+2}(x)\}$$

$$=\lim_{n=\infty}\frac{2}{3}\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{dx}{1+x}P_n(x)$$

But

$$I\equiv\lim_{n=\infty}\frac{2}{3}\left|\int_{\cos\epsilon}^{1}\frac{F_1(x)}{(1-x)^2}\;\frac{dx}{1+x}\;P_n(x)\right|$$

$$\leq \frac{2}{3} \int_{\cos \epsilon}^{1} \left| \frac{F_1(x)}{(1-x)^2} \quad \frac{1}{1+x} \right| dx$$

$$\leq \frac{2(1-\cos \epsilon)}{3} \times M_1,$$

where M_1 is a finite quantity since the function is summable in the interval. Hence,

$$I \leq \frac{4M_1}{3} \sin^2 \frac{\epsilon}{2}$$

$$\leq \frac{M_1 \epsilon^2}{3} < \epsilon \text{ if } \epsilon \text{ is chosen sufficiently small}$$

Therefore the left-hand side in equation (9) is less than ϵ

2. The expansion of any arbitrary function $f(x)$ in a Legendre series is given by

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + . \quad + a_n P_n(x) + \quad ,$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(t) \, P_n(t) dt,$$

provided the expansion is valid

Then

$$f(x) = \tfrac{1}{2} \sum_{n=0}^{\infty} (2n+1) \int_{-1}^{1} f(t) \, P_n(t) \, P_n(x) dt$$

and

$$f(1) = \tfrac{1}{2} \sum_{n=0}^{\infty} (2n+1) \int_{-1}^{1} f(t) P_n(t) \, dt.$$

The arithmetic mean of the first $(n+1)$ terms of the series

$$\sum_{m=0}^{n} a_m,$$

that is,

$$\tfrac{1}{2}\sum_{m=0}^{n}\int_{-1}^{1} f(t).(2m+1)P_m(t)\,dt$$

is, where $S_n^{(1)}(t)$ has the same meaning as in § 1,

$$\tfrac{1}{2}\int_{-1}^{1} f(t)\,S_n^{(1)}(t)\,dt$$

$$=-\tfrac{1}{2}\frac{1}{n+1}\int_{-\pi}^{\pi} f(t)\Big\{P_o(t)\frac{\sin^2(n+1)\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+P_1(t)\frac{\sin^2 n\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

$$+\ldots+P_n(t)\frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\Big\}\sin\theta\,dt,\ \text{where } t=\cos\theta.$$

It will be readily seen that the consideration of the limit of this integral reduces to the consideration of the limit of

$$\frac{1}{n+1}\int_{0}^{\epsilon} f(\cos\theta)\sin\theta\,d\theta\,\Big\{P_0(\cos\theta)\frac{\sin^2(n+1)\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+$$

$$P_1(\cos\theta)\frac{\sin^2 n\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\ldots+P_n(\cos\theta)\frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\Big\},$$

where ϵ is arbitrarily small, but fixed.

Integrating by parts, and denoting

$$\int_{0}^{\theta} f(\cos\theta)\sin\theta\,d\theta_2$$

by F (θ), we have the given integral

$$=\frac{1}{n+1}\Big[\mathrm{F}(\theta)\,\mathrm{S}_n^{(1)}\,(\cos\theta)\Big]_0^{\epsilon}$$

$$+\frac{1}{n+1}\int_0^{\epsilon}\mathrm{F}(\theta)\,\cot\frac{\theta}{2}\,\mathrm{S}_n^{(1)}(\cos\theta)\,d\theta$$

$$+\frac{2}{n+1}\int_0^{\epsilon}\frac{\mathrm{F}(\theta)}{1-\cos\theta}\,\sin\theta\,d\theta\left\{\frac{d\mathrm{P}_1}{dx}\sin^2\frac{n\theta}{2}\right.$$

$$\left.+\frac{d\mathrm{P}_2}{dx}\sin^2\frac{(n-1)\theta}{2}+\quad+\frac{d\mathrm{P}_n}{dx}\sin^2\frac{\theta}{2}\right\}$$

$$-\frac{2}{n+1}\int_0^{\epsilon}\frac{\mathrm{F}(\theta)}{1-\cos\theta}\;\{(n+1)\sin(n+1)\theta\,\mathrm{P}_0(\cos\theta)$$

$$+\,n\sin n\theta\mathrm{P}_1(\cos\theta)+\quad+\mathrm{P}_n\sin\theta\}d\theta. \tag{10}$$

3 For our *first example*, we put

$$f\,(\cos\theta)=\cos\frac{1}{1-\cos\theta}\quad,$$

so that $\mathrm{F}(\theta)=\underline{\Phi}(\theta)$ in corollary 1 of Lemma I. Corresponding to this function equation (10) gives us four terms on the right hand side. Of these, the first term is, on account of Lemma 2, less in absolute value than $2\epsilon^2$; the second term is less in absolute value than 4ϵ, by Lemma 3 ; the third term is less in absolute value than $\mathrm{C}'\epsilon$, by Lemma 4 ; and the last is less in absolute value than $\mathrm{M}\epsilon^2$, where M is a finite constant by Lemma 6. Thus the limiting value of the arithmetic mean of the first $(n+1)$ terms of the series Σa_n, corresponding to the function $\cos\dfrac{1}{1-\cos\theta}$ can be made as small as we like by choosing ϵ, sufficiently small and then making n infinite. Hence at the origin, the Legendre series corresponding to $\cos\dfrac{1}{1-\cos\theta}$ is summable (C, 1), the sum being zero.

Nevertheless, *Fejér's condition* that

$$\lim_{\epsilon=0} \frac{1}{K_\epsilon} \int\int_{K_\epsilon} \left| \cos \frac{1}{1-\cos\theta} \right| do$$

(where $K_\epsilon = 4\pi \sin^2 \frac{\epsilon}{2}$, and $do = \sin\theta \, d\theta \, d\phi$)

$$= \lim_{\epsilon=0} \frac{1}{4\pi \sin^2 \frac{\epsilon}{2}} \int_0^{2\pi} d\phi \int_0^{\epsilon} \left| \cos \frac{1}{1-\cos\theta} \right| \sin\theta \, d\theta$$

$$= \lim_{\epsilon=0} \frac{1}{1-\cos\epsilon} \int_0^{\epsilon} \left| \cos \frac{1}{1-\cos\theta} \right| \sin\theta \, d\theta$$

$$= \lim_{\epsilon=0} \frac{1}{1-\cos\epsilon} \int_0^{1-\cos\epsilon} \left| \cos \frac{1}{t} \right| dt,$$

should be equal to zero, is *not* satisfied, since it is known* that

$$\lim_{z=0} \frac{1}{z} \int_0^{z} \left| \cos \frac{1}{t} \right| dt = \frac{2}{\pi}$$

4. For the *second example*, we take

$$f(\cos\theta) = \frac{1}{(1-\cos\theta)^{\frac{1}{2}}} \cos \frac{1}{(1-\cos\theta)^{\frac{3}{2}}},$$

so that $f(\cos\theta)$ has an *infinite* discontinuity of the second kind at $\theta=0$, although it is absolutely integrable In this case, $F_-(\theta) = \Phi(\theta)$ in Corollary 2 of Lemma 1, m being equal to $\frac{1}{2}$ and n equal to $\frac{3}{2}$. As

* G Prasad—*Recent researches in the Theory of Fourier Series* (Calcutta, 1928) pp. 64-67.

in the first example, we can prove that the four terms on the right hand side of equation (10), can each be made as small as we like by choosing ϵ, sufficiently small and making n infinite. Thus, in this case too, the limiting value of the arithmetic mean of the Legendre series corresponding to

$$\frac{1}{(1-\cos\theta)^{\frac{1}{2}}}\cos\frac{1}{(1-\cos\theta)^{\frac{3}{2}}}$$

at the origin can be made as small as we like, by suitably choosing ϵ, and consequently the Legendre series corresponding to this function is summable (C, 1) at the origin, this sum being also zero

Fejér's condition in this case, that

$$\lim_{\epsilon=0}\frac{1}{K_\epsilon}\iint_{K_\epsilon}\left|\frac{1}{(1-\cos\ \theta)^{\frac{1}{2}}}\cos\frac{1}{(1-\cos\ \theta)^{\frac{3}{2}}}\right|d\sigma$$

$$=\lim_{\epsilon=0}\frac{1}{1-\cos\ \epsilon}\int_0^{1-\cos\ \epsilon}\left|\frac{1}{\sqrt{t}}\cos\frac{1}{t^{\frac{3}{2}}}\right|dt,$$

should be equal to zero, is also *not* satisfied, since it is known* that

$$\lim_{z=0}\frac{1}{z}\int_0^z\left|\frac{1}{\sqrt{t}}\cos\frac{1}{t^{\frac{3}{2}}}\right|dt,$$

is infinite

* *G Prasad*— On the failure of Lebesgue's criterion for the summability (C, 1) of the Fourier Series of a function at a point where the function has a discontinuity of the second kind" (*Bulletin of the Calcutta Mathematical Society*, Vol XIX, 1928, p 8)

1

ON A TYPE OF MODULAR RELATION

BY

S. C. MITRA

The object of the present paper is to establish several identities relating to theta functions It is believed that the results obtained by me are new

1 Let

$$\mu = \frac{q^{\frac{9}{20}}(1+q^2)(1+q^8)(1+q^{12})(1+q^{18})\ldots}{q^{\frac{1}{20}}(1+q^4)(1+q^6)(1+q^{14})(1+q^{16})\ldots\ldots} \quad \ldots \quad (1)$$

The result of replacing q by q^p in (1) will be written μ_p, while a dash attached to μ will denote a like function of the complementary modulus q'

2 From the formula *

$$\sqrt{w}\, \mathrm{I}_3(x, q) = \sum_{-\infty}^{\infty} e^{-\frac{\pi}{w}\left(\frac{x}{\pi}+n\right)^2}$$

where $e^{-\pi w} = q$, we have

$$\frac{\mathrm{I}_3\left(\frac{3\pi}{10}, q^{\frac{1}{5}}\right)}{\mathrm{I}_3\left(\frac{\pi}{10}, q^{\frac{1}{5}}\right)} = \frac{\sum e^{-\frac{5\pi}{w}\left(n+\frac{3}{10}\right)^2}}{\sum e^{-\frac{5\pi}{w}\left(n+\frac{1}{10}\right)^2}} = \mu' \quad \ldots \quad (2)$$

Let us write u_n for $\mathrm{I}_3\left(\frac{\pi}{10}n, q\right)$ where

$n=0, 1, 2, 3, 4$ and 5

* Tannery et Molk, Fonctions Elliptiques, p 264.

We have the formula

$$I_3(x+y+z)I_3(x)I_3(y)I_3(z)+I_4(x+y+z)I_4(x)I_4(y)I_4(z)$$

$$=I_3(0)I_3(x+y)I_3(y+z)I_3(z+x)+I_4(0)I_4(x+y)I_4(y+z)I_4(z+x) \qquad (3)$$

Let us put $x=y=\frac{\pi}{10}$, $z=\frac{\pi}{5}$.

We have

$$u_1u_3u_4^2+u_1^2u_2u_4=u_2^2u_3u_5+u_0u_2u_3^2 \qquad . \quad (4)$$

Again when we put

$$x=y=\frac{3\pi}{10}, \quad z=\frac{3\pi}{5}$$

we get

$$u_1u_2^2u_3+u_2u_3^2u_4=u_1u_4^2u_5+u_0u_1^2u_4 \qquad .. \quad (5)$$

From a known formula for $I_4(2x, q^2)$, we have

$$\frac{I_3\left(\frac{3\pi}{10}, q^{\frac{2}{5}}\right)}{I_3\left(\frac{\pi}{10}, q^{\frac{2}{5}}\right)}=\frac{u_1}{u_3}\ \frac{u_4}{u_2}$$

But $\frac{u_3}{u_1}=\mu'$ and (2) gives $\frac{u_4}{u_2}=\mu'\mu'_{\frac{1}{2}}$.

Therefore from (4) and (5) we get

$$\frac{u_0}{u_2}=\frac{(1+\mu'^2\mu'_{\frac{1}{2}})(1-\mu'\mu'^3_{\frac{1}{2}})}{\mu'_{\frac{1}{2}}(1-\mu'^2\mu'_{\frac{1}{2}})} \qquad .. \quad (6)$$

and

$$\frac{u_5}{u_3}=-\frac{(\mu'-\mu'^2_{\frac{1}{2}})(1+\mu'^2\mu'_{\frac{1}{2}})}{\mu'\mu'_{\frac{1}{2}}\ (1-\mu'^2\mu'_{\frac{1}{2}})} \qquad ... \quad (7)$$

Again let us put $x=\frac{\pi}{10}$, $y=\frac{2\pi}{10}$ and $z=\frac{3\pi}{10}$ in (3)

we get

$$2\,u_1u_2u_3u_4=u_0u_5\,(u_1u_2+u_3u_4) \qquad . \quad (8)$$

Let ν stand for the expression

$$\frac{q^{\frac{9}{20}}(1-q^2)(1-q^8)(1-q^{12})(1-q^{18})}{q^{\frac{1}{20}}(1-q^4)(1-q^6)(1-q^{14})(1-q^{16})} \qquad (9)$$

Let

$$\frac{I_2\left(\frac{3\pi}{10}, q^{\frac{1}{5}}\right)}{I_2\left(\frac{\pi}{10}, q^{\frac{1}{5}}\right)} = \frac{\sum(-1)^n e^{-\frac{5\pi}{w}\left(n+\frac{3}{10}\right)^2}}{\sum(-1)^n e^{-\frac{\pi}{w}\left(n+\frac{1}{10}\right)^2}} = \nu' \qquad \ldots (10)$$

From a well-known formula for $I_1(2x, q^2)$ we have

$$\frac{I_2\left(\frac{4\pi}{10}, q^{\frac{1}{5}}\right)}{I_2\left(\frac{2\pi}{10}, q^{\frac{1}{5}}\right)} = \nu'\nu'_{\frac{1}{2}}$$

In the identity

$$I_3(0)I_4(0)I_1(\nu+c)I_2(\nu-c)$$

$$= I_3(c)I_4(c)I_1(\nu)I_2(\nu)+I_1(c)I_2(c)I_3(\nu)I_4(\nu), \qquad \ldots (11)$$

let us put $c=\frac{\pi}{10}$, $\nu=\frac{2\pi}{10}$ We get

$$u_0u_5 = u_1u_4\ \nu' + u_2u_3\ \nu'\nu'_{\frac{1}{2}} \qquad \ldots (12)$$

Therefore from (8) and (11) we have

$$\frac{2u_1u_4}{u_1u_2+u_3u_4} = \mu'_{\frac{1}{2}}\ \nu' + \nu'\nu'_{\frac{1}{2}}\ .$$

or

$$\frac{2\mu'\mu'_{\frac{1}{2}}}{1+\mu'^2\mu'_{\frac{1}{2}}} = \mu'_{\frac{1}{2}}\ \nu' + \nu'\nu'_{\frac{1}{2}} \qquad \ldots (13)$$

Now $\mu' = \frac{\nu_2'}{\nu'}$.

After simplification we get

$$2\nu'_2\nu'_{\frac{1}{2}}=(\nu'+\nu'^2_{\frac{1}{2}})(\nu'^2_2+\nu'\nu'_{\frac{1}{2}})$$

Suppressing dashes and changing $\nu_{\frac{1}{2}}$ and ν into ν and ν_2 respectively, we get the identity

$$2\nu_4\nu=(\nu_2+\nu^2)(\nu_4^2+\nu_2\nu) \qquad \ldots \quad (A)$$

Ramanujan * has proved that the relation between ν and ν_2 is

$$\nu^2+\nu\nu_2^3+\nu^3\nu_2^2-\nu_2=0 \qquad \ldots \quad (14)$$

Eliminating ν_2 between (14) and (A) we get the biquadratic relation

$$(\nu_4^6\nu+\nu_4\nu^6)-(\nu_4^5+\nu^5)+\nu_4^5\nu^5-5\nu_4^4\nu^4$$
$$+10\nu_4^3\nu^3-5\nu_4^2\nu^2+\nu_4\nu=0 \qquad \ldots \quad (B)$$

a result which, I believe, has not been given by any previous writer.

2 Let

$$y=\frac{I_3\left(\frac{\pi}{10}, q^{\frac{1}{5}}\right)}{I_3\left(\frac{2\pi}{10}, q^{\frac{1}{5}}\right)}.$$

We have the formulæ

$$2I_3(2x, q^4)=I_3(x, q)+I_4(x, q)$$
$$2I_2(2x, q^4)=I_3(x, q)-I_4(x, q) \qquad \ldots \quad (15)$$

In the first of these two formulæ, let us put $x=\frac{\pi}{10}$ and $\frac{3\pi}{10}$ in succession. We have

$$\frac{I_3\left(\frac{4\pi}{10}, q^4\right)}{I_3\left(\frac{2\pi}{10}, q^4\right)}=\frac{I_3\left(\frac{3\pi}{10}\right)+I_3\left(\frac{2\pi}{10}\right)}{I_3\left(\frac{\pi}{10}\right)+I_3\left(\frac{4\pi}{10}\right)}, \qquad \ldots \quad (16)$$

* *Proc. L.M.S*, Vol. XIX, Series 2

whence we get

$$y=(\mu'\mu'_{\frac{1}{2}}\mu'_{\frac{1}{4}}\mu'_{\frac{1}{8}}-1)/(\mu'-\mu'_{\frac{1}{4}}\mu'_{\frac{1}{8}})$$

$$=\nu'(\nu'_2-\nu'_{\frac{1}{8}})/(\nu'_2\,\nu'_{\frac{1}{8}}-\nu'\nu'_{\frac{1}{2}}) \qquad . \quad (17)$$

From the second formula, we get

$$\frac{I_2\left(\frac{4\pi}{10},\, q^4\right)}{I_2\left(\frac{2\pi}{10},\, q^4\right)}=\frac{u_2-u_3}{u_1-u_4},$$

or

$$\nu'_{\frac{1}{4}}\nu'_{\frac{1}{8}}=\frac{\nu'_{\frac{1}{2}}(\nu'-\nu'_2 y)}{\nu'(\nu'_{\frac{1}{2}}y-\nu'_2)}$$

Substituting for y from (17), we get

$$\nu'_{\frac{1}{4}}\nu'_{\frac{1}{8}}=\frac{\nu'_{\frac{1}{2}}(2\nu'_2\nu'_{\frac{1}{8}}-\nu'^2_2-\nu'\nu'_{\frac{1}{2}})}{(2\nu'_2\nu'\nu'_{\frac{1}{2}}-\nu'\nu'_{\frac{1}{8}}\nu'_{\frac{1}{2}}-\nu'^2_2\nu'_{\frac{1}{8}})}$$

Suppressing dashes and changing $\nu_{\frac{1}{8}}$, $\nu_{\frac{1}{4}}$, $\nu_{\frac{1}{2}}$ and ν into ν, ν_2, ν_4 and ν_8 respectively, we get the relation

$$2\ \nu_{16}\nu_4\nu(1-\nu_8\nu_2)=(\nu_{16}^2+\nu_8\nu_4)(\nu_4-\nu_2\nu^2) \qquad \ldots \quad (C)$$

From (A) and (C), we get

$$(\nu_4-\nu_2\nu^2)=\nu(1-\nu_8\nu_2)(\nu_8+\nu_4^2) \qquad . \quad (D)$$

3 We have the formulæ

$$I_3(2x,\, q^2)=\frac{I_3^2(x)+I_4^2(x)}{2I_3(0,\, q^2)}$$

$$I_2(2x,\, q^2)=\frac{I_3^2(x)-I_4^2(x)}{2I_2(0,\, q^2)}\,. \qquad \ldots \quad (18)$$

Putting $x=\frac{2\pi}{10}$ and $\frac{\pi}{10}$ in succession, we have from the former of these two identities

$$\frac{I_3\left(\frac{4\pi}{10},\, q^2\right)}{I_3\left(\frac{2\pi}{10},\, q^2\right)}=\frac{I_3^2\left(\frac{2\pi}{10}\right)+I_3^2\left(\frac{3\pi}{10}\right)}{I_3^2\left(\frac{\pi}{10}\right)+I_3^2\left(\frac{4\pi}{10}\right)} \qquad \ldots \quad (19)$$

whence we get

$$y^4=\frac{\nu'^2(\nu'_2{}^2\nu'-\nu'_{\frac{1}{2}}{}^2\nu'_{\frac{1}{4}})}{\nu'_{\frac{1}{2}}{}^2(\nu'_2{}^2\nu'_{\frac{1}{4}}-\nu'^3)} \cdot$$

From the second identity we get

$$y^2=\frac{\nu'^2\ (\nu'_2{}^2\nu'_{\frac{1}{4}}+\nu'_{\frac{1}{2}})}{\nu'_{\frac{1}{2}}\ (\nu'_2{}^2+\nu'^2\nu'_{\frac{1}{2}}\nu'_{\frac{1}{4}})} \cdot$$

Therefore we have

$$2\nu'_2{}^2\nu'_{\frac{1}{2}}\nu'_{\frac{1}{4}}\ (\nu'_{\frac{1}{2}}-\nu'^3)=(\nu'^2\nu'_{\frac{1}{2}}{}^2+\nu'_2{}^4)(\nu'-\nu'_{\frac{1}{2}}\ \nu'_{\frac{1}{4}}{}^2)$$

Suppressing dashes and changing $\nu_{\frac{1}{4}}$, $\nu_{\frac{1}{2}}$ into ν and ν_2 respectively, we get the identity

$$2\nu_8{}^2\nu_2\nu(\nu_2-\nu_4{}^3)=(\nu_4{}^2\nu_2{}^2+\nu_8{}^4)(\nu_4-\nu_2\nu^2) \qquad .. \quad \text{(E)}$$

My best thanks are due to Dr Ganesh Prasad who kindly suggested the investigation to me and took great interest in the preparation of the paper.

Bull Cal Math. Soc, Vol XXI, No 2, (1929)

2

TABLE OF COMPLEX MULTIPLICATION MODULI

BY

S. C. MITRA.

$\Delta=235$

$$t=\sqrt[4]{4\kappa\kappa'}$$

$$t^3-(7+4\sqrt{5})t^2+(103+46\sqrt{5})\,t-1=0$$

$\Delta=355$

$$\alpha=\frac{(1-t^8)^3}{t^8}\,,$$

$$\alpha^2+A\alpha+B=0,$$

$$A=-(99437516760419419104000$$
$$+44469809398014498092160\sqrt{5})$$

$$B=(53498193625276219441152000$$
$$+23925119523912707604480000\sqrt{5})$$

$\Delta=203$

$$S=\sqrt[12]{\frac{\kappa\kappa'}{4}}$$

$$16s^{12}-16s^{11}+32s^{10}-56s^8-8s^7+40s^6+28s^5-32s^4$$
$$-16s^3+26s^2-10s+1=0$$

$\Delta=179$

$$32s^{15}-32s^{13}+16s^{12}+96s^{11}+176s^{10}+160s^{9}+64s^{8}-8s^{7}-16s^{6}$$
$$+32s^{5}+76s^{4}+62s^{3}+20s^{2}-1=0$$

$$\Delta=118$$

$$\gamma=\sqrt{2}\left[\left\{\frac{1}{2}\left(k^{-1}-k\right)\right\}^{\frac{1}{2}}+\left\{\frac{1}{2}\left(k^{-1}-k\right)\right\}^{-\frac{1}{2}}\right],$$

$$\gamma^{3}-2540\gamma^{2}+9392\gamma-9280=0$$

$$\Delta=139$$

$$8s^{9}+8s^{8}+16s^{7}+28s^{6}+16s^{5}+4s^{4}+10s^{3}+10s^{2}+2s-1=0$$

$$\Delta=155$$

$$16s^{12}+80s^{11}+160s^{10}+176s^{9}+136s^{8}+72s^{7}-8s^{6}-52s^{5}-40s^{4}$$
$$-20s^{3}-10s^{2}-2s+1=0$$

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3

On sound waves due to prescribed vibrations of a cylindrical surface in the presence of a rigid and fixed cylindrical obstacle

Part II.

BY

Hrishikesh Sircar

(*University of Dacca*)

In a previous communication,* under the same title, we have discussed sound-waves due to prescribed vibrations on the surface of a right circular cylinder in the presence of another right circular cylinder which is rigid and fixed. In the present paper I have attempted to consider sound-waves due to prescribed vibration on the surface of a right circular cylinder in the presence of a rigid and fixed elliptic cylinder of small eccentricity The success depends upon the transformation-theorems employed in the paper already cited, and upon the theory, as developed by H. Poincare,† Helge Von Koch and others, of solving linear equations, when the unknown quantities as well as the equations to determine them are infinite in number.

I.

Let a_1 denote the radius of the right circular cylinder and a and b, the semi-axes of the fixed elliptic cylinder. Let D denote the distance between their axes which we suppose to be parallel If we suppose that the two cylinders are infinitely long and the prescribed vibration is transverse to the axis of the vibrating cylinder, the problem would be a two-dimensional one

* N M. Basu and H. Sircar, *Bull Cal. Math. Soc.*, Vol. XVIII, No. 2.

† Remarques sur l'emploi de la méthode precéndenté,—*Bulletin de la Société Mathematique de France*, T 13, p. 19. Sur les déterminants d'ordre infini—*Bulletin de la Société Mathematique de France*, T. 14, p. 77. *Acta Mathematica*, Vols. 15 and 16.

The equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, can be written as

$$r=a\left[1-\epsilon+\epsilon\cos 2\theta\right], \text{ where } \epsilon=\frac{e^2}{4}, \quad \text{and}$$

higher powers of e beyond e^2 have been ignored

Let (ξ, η) denote the elliptic co-ordinates of a point of which the Cartesian and polar co-ordinates are respectively (x', y') and (r', θ'), referred to the centre of the ellipse as origin Then, we have,

$$x'=r'\cos\theta'=k_0\cosh\xi\cos\eta=k_0\zeta\mu_1,$$

and $$y'=r'\sin\theta'=k_0\sinh\xi\sin\eta=k_0\sqrt{\eta^2-1}\sqrt{1-\mu_1^2},$$

where $$\zeta=\cosh\xi,$$

$$\mu_1=\cos\eta,$$

and $$k_0=a'e'=ae,$$

a' and e' denoting respectively the semi-major-axis and eccentricity of the confocal ellipse passing through the point under consideration

Therefore,

$$r'^2=k_0^2(\zeta^2+\mu_1^2-1)$$

and $$\tan\theta'=\frac{\sqrt{1-\mu'^2}}{\mu'}=\frac{\sqrt{\zeta^2-1}\sqrt{1-\mu_1^2}}{\mu_1\zeta}, \quad (\mu'=\cos\theta'),$$

whence

$$\left.\begin{aligned}\frac{\partial r'}{\partial\zeta}&=k_0^2\frac{\zeta}{r'}\\ \text{and}\quad \frac{\partial\mu'}{\partial\zeta}&=\frac{\mu'(1-\mu'^2)}{\zeta(1-\zeta^2)}\end{aligned}\right\} \qquad (1)$$

II

The prescribed normal vibration at any point on the surface of the vibrating cylinder can be developed into a Fourier series and we accordingly assume, for the normal vibration, an expression of the form,

$$\sum_{n=0}^{n=\infty}(U_n\cos n\theta+V_n\sin n\theta)e^{ikct}$$

Let ϕ denote the velocity potential of the sound waves. Then ϕ must satisfy the following equations:—

(i) $$\ddot{\phi}=c^2\nabla_1^2\phi,$$

at all points in the surrounding medium, where

$$\nabla_1^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

or

$$\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2}{\partial \theta'^2}$$

(ii) $$\frac{\partial \phi}{\partial r}=-\sum_{n=0}^{n=\infty}(U_n\cos n\theta+V_n\sin n\theta)e^{ikct}\quad\text{on } r=a_1$$

and

(iii) $$\frac{\partial \phi}{\partial n}=0,\qquad\text{on the ellipse.}$$

Where dn denotes an element of normal at any point on the boundary of the elliptic section by a transverse plane.

Let us, first of all, suppose that the vibrating right circular cylinder is outside the fixed elliptic cylinder, the centre lying on the major axis at a distance D. Let (r, θ) and (r', θ') denote the polar co-ordinates of the same point, referred to the centres of the vibrating and fixed cylinders respectively, θ and θ' being measured in opposite senses from the major-axis.

III

Let

$$\phi=\psi e^{ikct},$$

ψ being a function of r and θ only.

The equation

$$\ddot{\phi}=c^2\nabla_1^2\phi$$

is then reduced to

$$(\nabla_1^2+k^2)\psi=0$$

Whatever be the nature of ψ as a function of r and θ, it can be developed in a Fourier series of the form

$$\sum_{n=0}^{n=\infty} (A_n \cos n\theta + B_n \sin n\theta)\psi_n,$$

where ψ_n is a function of r only and satisfies the differential equation

$$\frac{\partial^2 \psi_n}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_n}{\partial r} + \left(k^2 - \frac{n^2}{r^2}\right)\psi_n = 0 \qquad . \qquad (\alpha)$$

The two boundary conditions on the two cylinders are reduced to

$$\frac{\partial \psi}{\partial r} = -\Sigma(U_n \cos n\theta + V_n \sin n\theta), \qquad \text{on} \quad r = a_1 \quad \text{and}$$

$$\frac{\partial \psi}{\partial \zeta} = 0, \quad \text{on the elliptic section,}$$

since $\qquad dn = \frac{d\zeta}{h}$ where

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2.$$

IV.

Noticing that ψ_n would represent a divergent system of waves, the appropriate solution of (α) would be

$$\psi_n = H_n^2 (kr)^*$$

Thus, the initial unobstructed wave-system diffusing itself outwards into infinite space would be represented by

$$\psi^0 = \sum_{n=0}^{n=\infty} (A_n \cos n\theta + B_n \sin n\theta) H_n(kr)$$

* H_n^2 has been regarded by Nielsen as standard solution of Bessel's equation and is described by him as function of the third kind This function occurs in Hankel's researches on integral representation and asymptotic expansions of $J_n(x)$ and $T_n(z)$ In honour of Hankel, Nielsen denotes it by H It is the 2nd of the functions of the third kind.

where

$$A_n = -\frac{U_n}{kH'_n(ka_1)},$$

$$B_n = -\frac{V_n}{kH'_n(ka_1)},$$

and H_n has been written for H_n^2 for the sake of convenience

But the value of $\psi(=\psi^0)$ does not satisfy the other boundary condition, and we accordingly assume,

$$\psi=\psi^0+\psi^1,$$

where ψ^1 represents the velocity-potential of the waves scattered from the fixed cylinder

ψ^1 must satisfy the two-dimensional wave-equation,

$$(\nabla_1^2+k^2)\psi^1=0,$$

and the condition $\dfrac{\partial}{\partial n}(\psi^0+\psi^1)=0$, on the fixed elliptic cylinder (β)

Remembering that ψ^1 must be of the nature of a divergent wave-system, we assume,

$$\psi^1=\sum_{m=-\infty}^{m=+\infty}(A'_m \cos m\theta'+B'_m \sin m\theta')H_m(kr')$$

Now

$$\psi^0=\sum_{n=0}^{n=\infty}(A_n \cos n\theta+B_n \sin n\theta)H_n(kr)$$

$$=\sum_{m=-\infty}^{m=\infty} J_m(kr')\Big[\cos m\theta' \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD)$$

$$+\sin m\theta' \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD)\Big],$$

since in the neighbourhood of the fixed cylinder

$$r'<D.$$

Hence from the boundary condition (β) on the fixed cylinder we have

$$0=\frac{\partial}{\partial\zeta}(\psi^0+\psi^1)=\frac{\partial}{\partial r'}(\psi^0+\psi^1)\frac{\partial r'}{\partial\zeta}+\frac{\partial}{\partial\theta'}(\psi^0+\psi^1)\frac{\partial\theta'}{\partial\zeta}$$

$$=\frac{\partial}{\partial r'}(\psi^0+\psi^1)\frac{k_0^2\zeta}{r'}-\frac{\sin\theta'\cos\theta'}{\zeta(1-\zeta^2)}\frac{\partial}{\partial\theta'}(\psi^0+\psi^1)$$

from (1) of section I

$$=\frac{k_0^2 k}{er}\sum_{m=-\infty}^{m=\infty}\Big[(A'_m\cos m\theta'+B'_m\sin m\theta')H'_m(kr')$$

$$+J'_m(kr')\Big\{\cos m\theta'\sum_{n=0}^{n=\infty}A_nH_{n+m}(kD)+\sin m\theta'\sum_{n=0}^{n=\infty}B_nH_{n+m}(kD)\Big\}\Big]$$

$$-\sin\theta'\cos\theta'\frac{e^3}{e^2-1}\sum_{m=-\infty}^{m=\infty}m\Big[H_m(kr')(-A'_m\sin m\theta'+B'_m\cos m\theta')$$

$$+J_m(kr')\Big\{-\sin m\theta'\sum_{n=0}^{n=\infty}A_nH_{n+m}(kD)+\cos m\theta'\sum_{n=0}^{n=\alpha}B_nH_{n+m}(kD)\Big\}\Big]$$

$$\text{on}\qquad r'=a(1-\epsilon+\epsilon\cos 2\theta'),$$

where we have made use of $\zeta=\dfrac{1}{e}$, on the ellipse,

or

$$0=ka^2\sum_{m=-\infty}^{m=\infty}\Big[\Big(A'_m\cos m\theta'+B'_m\sin m\theta'\Big)\Big\{H_{m-1}(kr')-H_{m+1}(kr')\Big\}$$

$$+\Big\{J_{m-1}(kr')-J_{m+1}(kr')\Big\}\Big\{\cos m\theta'\sum_{n=0}^{n=\infty}A_nH_{n+m}(kD)$$

$$+\sin m\theta'\sum_{n=0}^{n=\alpha}B_nH_{n+m}(kD)\Big\}\Big]$$

$$+8a\epsilon\sin\theta'\cos\theta'\sum_{m=-\infty}^{m=\infty}m\Big[(-A'_m\sin m\theta'+B'_m\cos m\theta')H_m(ka)$$

$$+J_m(ka)\Big\{-\sin m\theta'\sum_{n=0}^{n=\infty}A_nH_{n+m}(kD)+\cos m\theta'\sum_{n=0}^{n=\alpha}B_nH_{n+m}(kD)\Big\}\Big]$$

$$\text{on} \qquad r' = a(1-\epsilon+\epsilon\cos 2\theta'),$$

where we have ignored powers of e beyond e^2, and have made use of

$$\zeta = \frac{1}{e}, \quad \text{on the ellipse,}$$

$$h_0 = ae,$$

and the recurrence-formulæ * of the type,

$$J'_n(z) = \tfrac{1}{2}[J_{n-1}(z) - J_{n+1}(z)]$$

$$H'_n(z) = \tfrac{1}{2}[H_{n-1}(z) - H_{n+1}(z)].$$

or

$$0 = ka \sum_{m=-\infty}^{m=\infty} \Big[(A'_m \cos m\theta' + B'_m \sin m\theta') \Big\{ H'_m(ka) - ka\epsilon\, H''_m(ka)(1-\cos 2\theta') \Big\}$$

$$+ \Big\{ J'_m(ka) - ka\epsilon J''(ka)(1-\cos 2\theta') \Big\} \Big\{ \cos m\theta' \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD) + \sin m\theta' \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD) \Big\} \Big]$$

$$+\epsilon \sum_{m=-\infty}^{m=\infty} \Big[H_m(ka) \Big\{ -A'm \Big(\cos (m-2)\theta' - \cos (m+2)\theta' \Big) + B'_m \sin (m+2)\theta' - \sin (m-2)\theta' \Big) \Big\}$$

$$+ J_m(ka) \Big\{ -\sum_{n=0}^{n=a} A_n H_{n+m}(kD) \Big(\cos (m-2)\theta' - \cos (m+2)\theta' \Big)$$

$$+ \sum_{n=0}^{n=a} B_n H_{n+m}(kD) \Big(\sin (m+2)\theta' - \sin (m-2)\theta' \Big) \Big\} \Big]$$

$$= 2ka \sum_{m=-\infty}^{m=\infty} \cos m\theta' \Big[A'm \Big\{ H'_m(ka) - ka\epsilon\, H''_m(ka) \Big\}$$

$$+ \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD) \Big\{ J'_m(ka) - ka\epsilon J''_m(ka) \Big\} \Big]$$

* See Watson's Theory of Bessel Functions, pp 17 and 74.

$$+2ka \sum_{m=-n}^{m=\infty} \sin m\theta' \Big[B'_m \Big\{ H'_m(ka) - ka\epsilon H''_m(ka) \Big\}$$

$$+ \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD) \Big\{ J'_m(ka) - ka\epsilon J''_m(ka) \Big\} \Big]$$

$$+ \sum_{m=-\infty}^{m=\infty} \cos(m-2)\theta' \Big[k^2a^2\epsilon A'_m H''_m(ka) + ka \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD)$$

$$-2m\epsilon A'_m H_m(ka) - 2m\epsilon J_m(ka) \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD) \Big]$$

$$+ \sum_{m=-\infty}^{m=\infty} \cos(m+2)\theta' \Big[k^2a^2\epsilon A'_m H''_m(ka) + ka \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD)$$

$$+2m\epsilon A'_m H_m(ka) + 2m\epsilon J_m(ka) \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD) \Big]$$

$$+ \sum_{m=-\infty}^{m=\infty} \sin(m+2)\theta' \Big[k^2a^2\epsilon B'_m H''_m(ka) + ka \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD)$$

$$+2m\epsilon B'_m H_m(ka) + 2m\epsilon J_m(ka) \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD) \Big]$$

$$+ \sum_{m=-\infty}^{m=\infty} \sin(m-2)\theta' \Big[k^2a^2\epsilon B'_m H''_m(ka) + (ka) \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD)$$

$$-2m\epsilon B'_m H_m(ka) - 2m\epsilon J_m(ka) \sum_{n=0}^{n=\infty} B_n H_{n+m}(kD) \Big] \qquad \ldots (\gamma)$$

V

Multiplying the equation (γ) by $\cos m\theta'$ and integrating between 0 and 2π, we have,

$$0 = 2ka \Big[A'm \Big\{ H'_m(ka) - ka\epsilon H''(ka) \Big\}$$

$$+ \sum_{n=0}^{n=\infty} A_n H_{n+m}(kD) \Big\{ J'_m(ka) - ka\epsilon J''_m(ka) \Big\} \Big]$$

$$+\epsilon A'_{m+2}\left\{h^2a^2\,H''_{m-2}(ha)-2(m+2)\,H_{m+2}(ha)\right\}$$

$$+\sum_{n=0}^{n=\infty}A_n\,H_{n+m+2}(hD)\left\{ha-2(m+2)\epsilon\,J_{m+2}(ha)\right\}$$

$$+\epsilon A'_{n-2}\left\{h^2a^2H''_{m-2}(ha)+2(m-2)\,H_{m-2}(ha)\right\}$$

$$+\sum_{n=0}^{n=\infty}A_n\,H_{n+m-2}(hD)\left\{ha+2(m-2)\epsilon\,J_{m-2}(ha)\right\} \qquad (\delta)$$

Again multiplying by $\sin m\theta'$ and integrating between 0 and 2π, we have,

$$0=2haB'_m\left\{H'_m(ha)-ha\epsilon\,H''_m(ha)\right\}$$

$$+2ha\sum_{n=0}^{n=\infty}B_n\,H_{n+m}(hD)\left\{J'_m(ha)-ha\epsilon\,J''_m(ha)\right\}$$

$$+\epsilon B'_{m+2}\left\{h^2a^2\,H''_{m+2}(ha)-2(m+2)H_{m+2}(ha)\right\}$$

$$+\sum_{n=0}^{n=\infty}B_n\,H_{n+m+2}(hD)\left\{ha-2(m+2)\,J_{m+2}(ha)\right\}$$

$$+\epsilon\,B'_{m-2}\left\{h^2a^2\,H''_{m-2}(ha)+2(m-2)\,H_{m-2}(ha)\right\}$$

$$+\sum_{n=0}^{n=\infty}B_n\,H_{n+m-2}(hD)\left\{ha+2\epsilon(m-2)\,J_{m-2}(ha)\right\} \qquad (\epsilon)$$

We thus have two infinite sets of linear equations to determine the A^1's and B^1's Such a system of linear equations was for the first time studied by Hill* who with the object of integrating a certain differential equation of the second order was led to consider a determinant

* *Acta Mathematica*, Vol. 8, pp 1-36 Reprinted, with some additions, from a paper published at Cambridge, U S A (1877)

of infinite order, H Poincare' * rigorously demonstrated the properties of these determinants, originally pointed out by Hill H Poincare',† Helge Von Koch ‡ and others have developed the theory of a system of linear equations, when the unknown quantities and the equations to determine them are infinite in number The constants A's and B's may be therefore determinate and the scattered wave-system becomes known

VI

But

$$\psi=\psi^0+\psi^1,$$

would no longer satisfy the boundary condition on the vibrating cylinder and necessarily we introduce a function ψ^2 which represents the velocity-potential of the system of waves (divergent in nature) scattered by the vibrating cylinder Then ψ^2 besides being a solution of the two-dimensional wave-equation

$$(\nabla_1^2+k^2)\psi^2=0,$$

satisfies the condition

$$\frac{\partial}{\partial r}(\psi^1+\psi^2)=0, \quad \text{on } r=a_1 \qquad .. \qquad (a)$$

Let us assume

$$\psi^2=\sum_{p=\infty}^{p=\infty}[A_p^2 \cos p\theta+B_p^2 \sin p\theta]\, H_p\,(kr) \qquad .. \qquad (b)$$

Now

$$\psi^1=\sum_{m=-\infty}^{m=\infty}(A_m' \cos m\theta'+B_m' \sin m\theta')\, H_m\,(kr')$$

$$=\sum_{p=-\infty}^{p=\infty} J_p\,(kr)\Big[\cos p\theta \sum_{m=-\infty}^{m=\infty} A_m'\, H_{p+m}\,(kD)$$

$$+\sin p\theta \sum_{m=-\infty}^{m=\infty} B_m'\, H_{p+m}\,(kD)\Big],$$

since $r<D$, in the neighbourhood of the vibrating cylinder.

Therefore from the above boundary condition (a) section (VI), we have,

$$0=\sum_{p=-\infty}^{p=\infty}[A_p^2 \cos p\theta+B_p^2 \sin p\theta]\, H_p'\,(ka_1)$$

* *Loc cit.*
† *Loc cit*
‡ *Loc cit.*

$$+\sum_{p=-\infty}^{p=\infty} J_p'(ka_1)\Big[\cos p\theta \sum_{m=-\infty}^{m=\infty} A_m' H_{p+m}(kD)$$

$$+\sin p\theta \sum_{m=-\infty}^{m=\infty} B_m' H_{p+m}(kD)\Big],$$

Whence

$$A_p{}^2 = -\frac{J_p'(ka_1)}{H_p'(ka_1)} \sum_{m=-\infty}^{m=\infty} A_m' H_{p+m}(kD)$$

and

$$B_p{}^2 = -\frac{J_p'(ka_1)}{H_p'(ka_1)} \sum_{m=-\infty}^{m=\infty} B_m' H_{p+m}(kD)$$

Thus, all the co-efficients in ψ^2 can be found out and ψ^2 is determined

Proceeding in this way, we find,

$$\psi=\psi^0+\psi^1+\psi^2+ \quad .$$

where the ψ's with even numbers denote velocity-potentials of the waves reflected from the vibrating cylinder and the ψ's with odd numbers represent the velocity-potentials of the waves reflected from the elliptic cylinder

The velocity-potential of the motion is thus given by

$$\phi=(\psi^0+\psi^1+\psi^2+ \quad \ldots)e^{ikct} .$$

VII

Let us next suppose that the vibrating cylinder is placed inside the elliptic cylinder in the same manner as in the foregoing case, and if we use the same notations and symbols, the motion inside the space contemplated can be similarly determined by obtaining the successive wave-systems reflected alternately from the internal and external boundaries

Assuming, as before, that the prescribed vibration of the cylinder is expressible by a series of the type

$$\sum_{n=0}^{n=\infty} (U_n \cos n\theta + V_n \sin n\theta)e^{ikct}$$

The velocity potential of the initial unobstructed wave-system is given, as before, by

$$\phi_0 = \sum_{n=0}^{n=\infty} (A_n \cos n\theta + B_n \sin n\theta)\, H_n(hr) e^{ihct}$$

where

$$A_n = -\frac{V_n}{h H_n(ha_1)}$$

$$\text{and } B_n = -\frac{V_n}{h H_n'(ha_1)}$$

This wave-system on incidence on the outer cylinder would disturb its boundary-condition and be scattered. If ϕ_1 denote the velocity-potential of this scattered system, we must have,

$$\frac{\partial}{\partial \zeta}(\phi_0 + \phi_1) = 0 \quad \text{on the elliptic boundary} \qquad (iii)$$

If the medium within the elliptic cylinder were un-interrupted, the velocity-potential of the motion would be finite at the origin and we accordingly assume

$$\phi_1 = \sum_{p=-\infty}^{p=\infty} (A^1{}_p \cos p\theta' + B^1{}_p \sin p\theta')\, J_p(hr')\, e^{ikct},$$

where the A^1's and B^1's are unknown constants to be determined from the above boundary condition (iii), Section VII.

In the neighbourhood of the fixed cylinder,

$$r' > D,$$

we, therefore, have

$$\phi_0 = \sum_{n=0}^{n=\infty} (A_n \cos n\theta + B_n \sin n\theta)\, H_n(hr)\, e^{ikct}$$

$$= \sum_{m=-\infty}^{m=\infty} \sum_{n=0}^{n=\infty} (-)^n J_m(hD)\, H_{n+m}(hr') \left[A_n \cos (n+m)\theta' \right.$$

$$\left. - B_n \sin (n+m)\theta' \right]$$

$$=\sum_{p=-\infty}^{p=\infty} H_p(kr') \, [\cos p\theta' \sum_{n=0}^{n=\infty} (-)^n A_n J_{n-p} (kD)$$

$$- \sin p\theta' \sum_{n=0}^{n=\infty} (-)^n B_n J_{n-p} (kD)]$$

Hence from the above boundary condition (*iii*), we have

$$0=\frac{\partial}{\partial\zeta}\left[\sum_{p=-\infty}^{p=\infty} H_p (kr') \left\{\cos p\theta' \sum_{n=0}^{n=\infty} (-)^n A_n J_{n-p} (kD)\right.\right.$$

$$\left. - \sin p\theta' \sum_{n=0}^{n=\infty} (-)^n B_n J_{n-p} (kD) \right\}$$

$$\left. + \sum_{p=-\infty}^{p=\infty} (A^1{}_p \cos p\theta' + B^1{}_p \sin p\theta')\right]$$

whence, proceeding as in sections IV and V, the unknown co-efficients can be found out and the scattered system of waves would be known

VIII

The waves represented by ϕ_1 will after incidence on the vibrating cylinder be reflected Let ϕ_2 denote the velocity-potential of the second system of scattered waves Remembering that it would be of the nature of a divergent wave-system, we assume,

$$\phi_2=\sum_{s=-\infty}^{s=\infty} (A^2{}_s \cos s\theta + B^2{}_s \sin s\theta) \, H_s(kr) \, e^{ikct}$$

The boundary condition on the vibrating cylinder must not be disturbed, therefore ϕ_2 must satisfy the condition

$$\frac{\partial}{\partial r}(\phi_1+\phi_2)=0, \qquad \text{on } r=a_1.$$

Now, in the neighbourhood of the vibrating cylinder, r may be greater or less than D Therefore, we have,

$$\phi_1=\sum_{p=-\infty}^{p=\infty} (A^1{}_p \cos p\theta' + B^1{}_p \sin p\theta') \, J_p(kr') \, e^{ikct}$$

$$=\sum_{s=-\infty}^{s=\infty}\sum_{p=-\infty}^{p=\infty} J_{s+p}\ (kD)\ J_s(kr)[A^1{}_p \cos s\theta + B^1{}_p \sin s\theta]\ e^{ikct}$$

$$=\sum_{s=-\infty}^{s=\infty} J_s(kr)\Big[\cos s\theta \sum_{p=-\infty}^{p=\infty} A^1{}_p\ J_{p+s}\ (kD)$$

$$+\sin s\theta \sum_{p=-\infty}^{p=\infty} B^1{}_p\ J_{s+p}\ (kD)\ \Big]\ e^{ikct}$$

From the above boundary condition we, therefore, have

$$0=\frac{\partial}{\partial r}\Big[\sum_{s=-\infty}^{s=\infty} J_s(kr)\Big\{\cos s\theta \sum_{p=-\infty}^{p=\infty} A^1{}_p J_{p+s}\ (kD)$$

$$+\sin s\theta \sum_{p=\infty}^{p=\infty} B^1{}_p\, J_{s+p}\ (kD)\Big\}$$

$$+\sum_{s=-\infty}^{s=\infty} (A^2{}_s \cos s\theta + B^2{}_s \sin s\theta)\ H_s\ (kr)\Big].$$

whence the unknown co-efficients can be found out

Obtaining in this way the velocity-potentials of the successive reflected waves, the final velocity-potential of the motion would be the sum of all these including that of the initial unobstructed motion

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4

ON THE EQUATION OF STATE

BY

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Van der Waals' Equation of State connecting the pressure and the volume of a gas can be written in the general form as an infinite series—

$$p+\frac{a}{V^2}=\frac{RT}{V}\left[1+\phi_1\frac{b}{V}+\phi_2\left(\frac{b}{V}\right)^2+\phi_3\left(\frac{b}{V}\right)^3+\ ..\right]. \qquad (1)$$

where p, V, R, T, a and b have their usual meaning and ϕ_1, ϕ_2, ... are the numerical coefficients

Methods are given for the evaluation of the numerical coefficients ϕ_1, ϕ_2, ϕ_3, ...of the above equation The coefficient ϕ_1 was first calculated by Vander Waals him self and subsequently by others [1] and was found to be equal to one. Taking into account the overlapped volume of the "Deckungssphären" when there are collisions between the different molecules two at a time Jäger [2] and soon afterwards Boltzmann [3] with two different methods obtained the value of ϕ_2 as $\frac{5}{8}$

Among the recent works on the determination of ϕ_2 may be mentioned those of Keesom [4] and A F Core [5] who considered the molecules as rigid elastic spheroids and discussed the effect of eccentricity on the value of ϕ_2 Their value is, however, same as given above

[1] In the methods given by Planck (*Sit . d. Kgl Preuss Akad. d Wiss* 1908 S 633 647) and subsequently independently by M. N. Saha and S N Bose (*Phil Mag* 36, pp. 199 202 Aug 1918) with the help of Boltzmann's Entropy equation the ordinary Vander Waals gas equation is obtained in a logarithmic form.

[2] G Jäger, *Sitzungsber d. Wiener Math naturn . Klasse* IIa, 105, p 15, 1896

[3] L. Boltzmann, *Wien Sit. Ber* [2a] 105 (1896) S 695 *Wiss. Abh* 3, S. 547, und [b] S 152 *Vergl auch Enc* V 8, Art Boltzmann und Nabl Nr. 29

[4] W H Keesom, *Amsterdam proc.* Vol. XV. part 1, 1912. S. 240 256.

[5] A F. Core, *Phil. Mag* 46, pp 256 272, Aug. 1923

The calculation of the coefficient ϕ_3 is a bit more difficult as in this case one has to consider the overlapped volume of the "Deckungsspharen" when there are collisions of three molecules at a time The first unsuccessful attempt in this direction was made by Vander Waals [1] Later on Hr Van Laar [2] attempted to evaluate ϕ_3 Soon after the above work of Laar, Boltzmann [3] pointed out certain mistakes in his calculation of the overlapped volume of three "Deckungsspharen" Thus when the corrections are made the value of ϕ_3 becomes 2869 Later on in 1906 H Happel [4] extended one of the methods given by Boltzmann for calculation of ϕ_2 in his "Vorlessungen uber gas theorie Band II S 143-151" to determine ϕ_3 The value obtained by him $\phi_3=0\ 288$, is nearly same as that obtained before by Laar-Boltzmann

A complete historical account of the different attempts made to get the equation of state will be found in Encyklopadie der Mathematischen Wissenschaften **B**and V. **1.** Heft **1**-6 S 669-751

In the present paper it is proposed to show how the Equation of State can be obtained up to any degree of approximation directly from Gibbs Law of Canonical distribution It may be noted in conclusion that the first attempt in this direction has been made by Wassmuth [5] who has got only Vander Waals gas equation

$$p+\frac{a}{V^2}=\frac{RT}{V}\left(1+\frac{b}{V}\right).$$

We have from the canonical distribution of Gibbs

$$\rho=Ne^{\frac{\psi-U}{\Theta}} \tag{2}$$

where ρ is the density of phase points having energy U, N the total number of phase points and ψ, Θ are two constants

[1] J D Vander Waals, [e] okt 1898 S 160 (The writer regrets his failure to get first hand information about this work.)

[2] Hr Van Laar, *Archives du Musee Joyler ser*, 2 6 p 237, 1900 = *Amsterdam Akad Versl* Jan 1899, S 350

[3] L Boltzmann, *Amsterdam Ber* 1899 S 477-484 = Boltzmann's *Wissen Schaftliche Abhandlungen* III Band (1882-1905), S 658 664

[4] H Happel, Habilitation-schrift tubingen (Leipzig) 1906 = *Ann d phys.* (4) 21 (1906) S 342

[5] A. Wassmuth, *Akad Wiss. Wien. Ber.* 122. 2a. S 651-666, März 1913

We have from equation (2) on integrating over the whole $\{\gamma\}$ space

$$\int \rho \Delta\tau = N \int e^{\frac{\psi - U}{\Theta}} \Delta\tau = N \qquad (3)$$

$$\text{Therefore} \int e^{\frac{\psi - U}{\Theta}} \Delta\tau = 1 \qquad (4)$$

$$\text{or} \qquad e^{-\frac{\psi}{\Theta}} = \int e^{-\frac{U}{\Theta}} \Delta\tau \qquad (5)$$

Now if each phase point represents n molecules having same mass μ and having positional and momenta co-ordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots x_n, y_n, z_n$ and $\mu\dot{x}_1, \mu\dot{y}_1, \mu\dot{z}_1 \ldots\ldots \mu\dot{x}_n; \mu\dot{y}_n, \mu\dot{z}_n$ respectively, then the volume-element in the phase-space becomes

$$\Delta\tau = \mu^{3n} dx_1 dy_1 dz_1 \ldots\ldots dz_n d\dot{x}_1 \ldots\ldots d\dot{z}_n$$

Thus the equation (4) reduces to

$$e^{-\frac{\psi}{\Theta}} = \mu^{3n} \int \ldots\ldots \int e^{-\frac{U}{\Theta}} dx_1 \ldots\ldots dz_n d\dot{x}_1 \ldots\ldots d\dot{z}_n$$

$$\text{or} \qquad e^{-\frac{\psi}{\Theta}} = \mu^{3n} \int \ldots \ldots \int e^{-\frac{L}{\Theta}} d\dot{x}_1 \ldots\ldots d\dot{z}_n \int e^{-\frac{\Phi}{\Theta}} dx_1 \ldots dz_n \qquad (6)$$

where L and Φ represent kinetic and potential energy respectively

Since the first integral term is a function of velocity, it does not contain V, the total volume of the gas, with respect to which we shall have to differentiate the whole series later on And hence the above equation may be written in the form

$$e^{-\frac{\psi}{\Theta}} = C \int \ldots\ldots \int e^{-\frac{\Phi}{\Theta}} dx_1 \ldots\ldots dz_n \qquad \ldots (7)$$

where C represents a constant independent of V

Again the average value of Φ is given by the relation

$$\Phi = -\frac{an^2}{V}$$

where a is a characteristic gas constant.

So we have $e^{-\frac{\psi}{\Theta}} = \mathrm{C}e^{\frac{an^2}{\Theta \mathrm{V}}} \int . \quad \int dx_1 \quad .... \quad dz_n$

$$= \mathrm{C}\, e^{\frac{an^2}{\Theta \mathrm{V}}}\; k \qquad .. \quad (8)$$

or $$-\frac{\psi}{\Theta} = \log \mathrm{C} + \frac{an^2}{\Theta \mathrm{V}} + \log k \qquad (9)$$

where $$k = \int \ldots \ldots \int dx_1 \qquad dz_n \qquad . \quad (10)$$

Now to find out k, we should take into account the correction to be applied to V due to the finite size of the molecules Thus when there is a single molecule in the volume V, the available volume for the second molecule is not V but $\mathrm{V}-\beta$ where β is the volume of sphere, drawn round the centre of each molecule with radius equal to the molecular diameter σ Again when there are already two molecules in the volume, the available volume for the third molecule is $\mathrm{V}-2\beta$ and more exactly, considering the overlapping volume of the two spheres due to the probability of their centres being within a distance σ and 2σ from each other, is $\mathrm{V}-2\beta+\frac{17}{64}\, 2\frac{\beta^2}{\mathrm{V}}$ * Similarly when there are three molecules, the available volume for the fourth one is $\mathrm{V}-3\beta+\frac{17}{64}.6\,\frac{\beta^2}{\mathrm{V}}$ and again more accurately, on considering the probable over-lapping volume of the three "Deckungsspharen" at a time,

$$\mathrm{V}-3\beta+\frac{17}{64}.6\,\frac{\beta^2}{\mathrm{V}}+\left[\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right] 6\,\frac{\beta^3}{\mathrm{V}^2} \;\dagger$$

Thus considering collisions only up to three molecules at a time, the available volume for the nth molecule when $n-1$ molecules are already present in the volume V is

$$\mathrm{V}-(n-1)\beta+(n-1)(n-2)\frac{\beta^2}{\mathrm{V}}\cdot\frac{17}{64}$$
$$+(n-1)(n-2)(n-3)\frac{\beta^3}{\mathrm{V}^2}\left[\frac{2357}{8\times 6720}-\frac{2\times 0{\cdot}0968}{8}\right]$$

* Boltzmann, Gastheorie, S 167

† Laar-Boltzmann l c.

Or neglecting the difference between n, $n-1$, $n-2$ and $n-3$, since n is very large, the above volume becomes

$$V-n\beta+\frac{n^2\beta^2}{V}\frac{17}{64}+\frac{n^3\beta^3}{V^2}\left[\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right]^*$$

Hence we have

$$\iiint dx_1dy_1dz_1=V\iiint dx_2dy_2dz_2=V-\beta$$

$$\iiint dx_3dy_3dz_3=V-2\beta+\frac{17}{64}2\frac{\beta^2}{V}$$

$$\iiint dx_4dy_4dz_4=V-3\beta+\frac{17}{64}\frac{6\beta^2}{V}+\left[\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right]6\frac{\beta^3}{V^2}$$

$$\iiint dx_5dy_5dz_5=V-4\beta+\frac{17}{64}\frac{12\beta^2}{V}+\left[\frac{2357}{8\times 6720}\right.$$

$$\left.-\frac{2\times 0\,0958}{8}\right]24\frac{\beta^3}{V^2}$$

and so on

and thus

$$k=V^n\left[1-\frac{\beta}{V}\right]\left[1-\frac{2\beta}{V}+\frac{17}{64}\frac{2\beta^2}{V^2}\right]$$

$$\times\left[1-\frac{3\beta}{V}+\frac{17}{64}\frac{6\beta^2}{V^2}+\left(\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right)\frac{6\beta^3}{V^3}\right]\quad\cdots$$

$$\times\left[1-\frac{n\beta}{V}+\frac{n^2\beta^2}{V^2}\frac{17}{64}+\frac{n^3\beta^3}{V^3}\left[\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right)\right]\quad\cdots\quad(11)$$

Therefore we have

$$\log k=n\log V+\sum_{n=1}^{n=n}\log\left[1-\frac{n\beta}{V}+\frac{n^2\beta^2}{V^2}\frac{17}{64}\right.$$

$$\left.+\frac{n^3\beta^3}{V^3}\left(\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right)\right]$$

* Laar-Boltzmann *l. c.*

$$=n \log V-\sum_{n=1}^{n=n}\left[\frac{n\beta}{V}-\frac{17 n^2\beta^2}{64\, V^2}+\frac{n^2\beta^2}{2V^2}-\left(\frac{2357}{8\times 6720}-\frac{2\times 0\,0958}{8}\right)\frac{n^3\beta^3}{V^3}\right.$$

$$\left.-\frac{17}{64}\;\frac{n^3\beta^3}{V^3}+\frac{n^3\beta^3}{3V^3}\right]$$

higher order than $\frac{\beta^3}{V^3}$ being neglected.

Or carrying out the above summation, we get

$$\log k \;=n \log V-\frac{n^2\beta}{2V}-\frac{5}{64}\;\frac{n^3\beta^2}{V^2}-\frac{2570\,5520}{53760\times 4}\;\frac{n^4\beta^3}{V^3} \quad \ldots \quad (12)$$

Therefore from equation (9) we have

$$-\frac{\psi}{\Theta}=\log C+\frac{a n^2}{\Theta V}+n \log V-\frac{n^2}{2}\;\frac{\beta}{V}-\frac{5}{64}\;\frac{n^3\beta^2}{V^2}$$

$$-\frac{2570\,5520}{53760\times 4}\;\frac{n^4\beta^3}{V^3} \quad \ldots \quad (13)$$

Hence remembering, $p=-\frac{\partial \psi}{\partial V}$, we get

$$\frac{p}{\Theta}=-\frac{a n^2}{\Theta V^2}+\frac{n}{V}+\frac{n^2\beta}{2V^2}+\frac{5}{32}\;\frac{n^3\beta^2}{V^3}+\frac{3\times 2570\,5520}{53760\times 4}\;\frac{n^4\beta^3}{V^4} \quad \ldots \quad (14)$$

Or putting $\frac{n\beta}{2}=b$ and $\Theta=KT$ we we have finally

$$p+\frac{a n^2}{V^2}=\frac{NKT}{V}\left[1+\frac{b}{V}+\frac{5}{8}\;\frac{b^2}{V^2}+\cdot 28689\;\frac{b^3}{V^3}\right] \ldots \quad (15)$$

which gives $\phi_3 =$ 28689 in good agreement with the value obtained by Boltzmann ($\phi_3 =$ 2869)

Before concluding this paper we should like to discuss a special case of some interest in the following section.

III

If we neglect the overlapping of the "De[illegible]" [illegible] tion (11) takes the following simple form

$$k = V^{n}\left[1-\frac{\beta}{V}\right]\left[1-\frac{2\beta}{V}\right]\cdots\cdots\left[1-[illegible]\right]$$

Therefore neglecting difference between n and [illegible] 1

$$\log k = n\log V + \log\left[1-\frac{\beta}{V}\right] + \log\left[1-\frac{2\beta}{V}\right] + \cdots$$

$$+ \log\left[1 - [illegible]\right]$$

Thus from the equation (9) we have

$$-\frac{\psi}{\Theta} = \log C + \frac{an^{2}}{\Theta V} + n\log V + \log\left(1-\frac{\beta}{V}\right) + \log\left(1 - [illegible]\right) + \cdots$$

$$+ \log\left(1 - [illegible]\right)\cdots$$

Therefore using the relation $p = -\frac{\partial\psi}{\partial V}$ we have from [illegible]

$$\frac{p}{\Theta} = -\frac{an^{2}}{\Theta V^{2}} + \frac{n}{V} + \frac{\beta/V^{2}}{1-\beta/V} + \frac{2\beta/V^{2}}{1-2\beta/V} + \cdots\cdots + [illegible]$$

$$= \frac{-an^{2}}{\Theta V^{2}} + \frac{n}{V} + \frac{\beta}{V^{2}}\left[\left(1+\frac{\beta}{V}+\frac{\beta^{2}}{V^{2}}+\cdots\right)\right.$$

$$+ 2\left(1+\frac{2\beta}{V}+\frac{2^{2}\beta^{2}}{V^{2}}+\cdots\cdots\right)$$

$$+ 3\left(1+\frac{3\beta}{V}+\frac{3^{2}\beta^{2}}{V^{2}}+\cdots\cdots\right)+\cdots\cdots$$

$$\left. + n\left(1+\frac{n\beta}{V}+\frac{n^{2}\beta^{2}}{V^{2}}+\cdots\cdots\right)\right]$$

And summing up all the series we have

$$\frac{p}{\Theta} = -\frac{an^2}{\Theta V^2}+\left[\frac{n\beta}{\beta V}+\frac{\beta}{V^2}\frac{n^2}{2}+\frac{\beta^2}{V^3}\frac{n^3}{3}+\frac{\beta^3}{V^4}\frac{n^4}{4}+\ldots\right]$$

$$= -\frac{an^2}{\Theta V^2}+\frac{1}{\beta}\left[\frac{n\beta}{V}+\frac{1}{2}\left(\frac{n\beta}{V}\right)^2+\frac{1}{3}\left(\frac{n\beta}{V}\right)^3+\ldots\right]$$

Now since $\Theta = KT$ we get finally

$$p = -\frac{KT}{\beta}\log\left(1-\frac{n\beta}{V}\right)-\frac{an^2}{V^2} \qquad (19)$$

which is the Planck-Saha-Bose Equation of state referred to in the introduction

It may be noted in conclusion that the present method can be easily extended to calculate the higher coefficients in equation (1), provided the corresponding overlapping of the "Deckungssphären" is known

Finally I must accord my best thanks to Dr K C Kar for his kindly snggesting the problem and giving valuable advice in course of the work

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5

The Jaina School of Mathematics

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Introductory

The present article does not profess to be a complete account of the mathematical achievements of the Jainas Indeed the account it gives is far from being complete and has been rather desultory The excuse for still wishing for the publication of this article inspite of its admitted imperfection and other deficiencies may be shortly stated thus The writer who has only recently began collecting materials for a full and comprehensive account of the contribution by the Jaina scholars to the development of Hindu mathematics, will have to refrain from further prosecuting his project now at this preliminary stage of the investigation So he wishes to keep in print a brief record of the results obtained by his labour in the hope that it will probably save the future and more successful researcher at least of some amount of his labour Moreover, even within this short span of time, there have been discovered certain mathematical results which are not only highly interesting but are also considered very important for the history of Hindu mathematics. Hitherto the sources of our informations about the achievements of the Hindus in the science of mathematics were practically confined to a period, the upper limit of which can be put at 499 A D , the date of composition of the *Āryabhaṭīya* by Āryabhata (born 476 A D). The *Sūrya-siddhānta* is believed undoubtedly to be an older composition , but it has gone through so many recensions that it is not easy to assert without any fear of contradiction, how much of the original matters have been retained in its present redaction. Here we give for the first time certain facts which will undoubtedly shift back the upper limit by eight

centuries at least if not more. It is hoped that this article will inspire some enthusiastic workers in the history of Hindu mathematics to a more careful, diligent and exhaustive search in this fruitful field

Place of mathematics in the Jainism

The Jainas attach great importance to the culture of mathematics. Their religious literature is generally classified into four groups, called *anuyoga*, meaning "the exposition of the principle" (of Jainism) One of them is the *ganitānuyoga* or "the exposition of the principle of mathematics" required in the Jainism. The knowledge of *samkhyāna* (literally "the Science of numbers," meaning arithmetic) and *jyotisa* ("Astronomy") is stated to be one of the principal accomplishments of the Jaina priest [1] It should be noted that the necessity of the Jaina priest to learn mathematics arises by way of finding the proper time and place for the religious ceremonies.[2] The Jainas attribute to the founder of their religion a sound knowledge of those sciences.[3] According to them, a child should be taught "firstly writing, then arithmetic as most important" of the seventy-two sciences or arts (*śilpa*) [4]

Sources Ganita sāra-samgraha of Mahāvīra.

The only treatise on arithmetic by a Jaina scholar which is available at present is the *Ganita-sāra-samgraha* [5] of Mahāvīra (850).

[1] *Bhagabatī sūtra* with the commentary of Abhayadeva Sūri (c 1050) edited by Āgamodayasamiti of Mehesana, 1919, Sūtra 90, *Uttarādhyayana-sūtra*, English translation by H Jacobi, Oxford, 1895, Ch XXV 7, 8, 38

[2] Compare the remark of Śānticandra Gani (1595 A D) in the preface to his commentary on the *Jambudvīpaprajñapti* "शुद्धगणितसिद्धे प्रशस्ते काले गृहीतानि प्रशस्तफलानि स्युः, कालश्च ज्योतिश्चाराधीनः, स च जम्बुद्वीपादिक्षेत्राधीनव्यवस्थस्तेनायं कालापरपर्यायो गणितानुयोग ।"

[3] *Kalpasūtra* of Bhadrabāhu (c 350 B C), English translation by H. Jacobi, ii 10 (SBE, vol 32, p 221)

[4] *The Antagada dasāo and Anuttarovavāya dasāo*, English translation by L D Barnett, 1907, p. 30 Compare *Kalpasūtra, loc cit*, p 282, Sūtra 211.

It is noteworthy that in the Buddhist canonical literature also, arithmetic is regarded as the first and the noblest of the arts (*śilpa*) *Vinaya Pitaka*, ed. Oldenberg, vol iv, p 7, *Majjima Nikāya*, vol. I, p 85, *Cullaniddesa*, p 199, etc

[5] *Ganita-sāra samgraha* of Mahāvīra, edited with English translation by M Rangacarya, Madras, 1912 In future the reference to this book, if not otherwise stated, will be to the English translation.

There are also known two astronomical treatises, called the *Sūrya-prajñapti* and the *Candraprajñapti* There were certainly other mathematical treatises by the early Jaina scholars, which are now lost In the introductory chapter of his book, Mahāvīra has expressed his obligation to a great number of previous mathematicians from whose works he has drawn his inspiration [1]

"With the help of the accomplished holy sages, who are worthy to be worshipped by the lords of the world, and of their disciples and disciples disciples, who constitute the well-known jointed series of preceptors, I glean from the great ocean of the knowledge of numbers a little of its essence, in the manner in which gems are (picked up) from the sea, gold is from the stony rock and the pearl from the oyster shell, and give out according to the power of my intelligence, the *Sāra-samgraha*, a small work on arithmetic, which is (however) not small in value"

Again in the concluding lines of the same chapter, he observes [2]

"Thus the terminology is stated briefly by the great sages What still remains to be said should be learnt in detail from the *Āgamas* "

Thus we come to know of the existence of other works on mathematics which were considered as the *Āgamas* or "sacred classics." The very name itself of Mahāvira's treatise *Ganita-sāra-saṁgraha* (or "the Collection of the essence of mathematics") reveals the existence of other treatises We have more direct proof of this Mahāvīra has quoted a rule by a Jaina mathematician for the solution of a certain class of problems [3] "This is the solution of (this kind of) problems as propounded by the learned, and the rule (itsef) has been declared by the great Jaina." It should be noted that the author of the *Ganita-sāra samgraha* has always held the Great Mahāvīra, the founder of Jaina religion to have been a great mathematician [4]

[1] *Ganita sāra saṁgraha*, i. 17-19

[2] *Ibid*, i 70 In this Rangacarya's translation has been slightly altered to make it more literal

[3] *Ibid*, vi 154.

[4] Compare i. 2

Bhadrabāhu and his Samhitā

Bhadrabāhu (died 170 A V = 298 B C)[1], a very prominent personage in the history of the Jaina religion, who is reputed as the last of the *Śrutakevalin* (*i e* those who can reproduce from memory the whole of the voluminous canonical literature of the Jainas), is known to be the author of two astronomical works (1) a commentary on the *Sūryaprajñapti*, and (2) an original work called the *Bhadrabāhavī Samhitā* None of these works is available at present. The former has been mentioned by Malayagiri (c 1150) in the opening verses of his own commentary on the *Sūryaprajñapti* and in fact he has quoted a few lines from that work[2] A work of the name of the *Bhadrabāhavī Samhitā* was found by Buhler,[3] but its authenticity has been suspected by modern scholars on the ground that (1) it is of the same character as the other *Samhitās*, (2) it has not been mentioned by Varāhamihira (505 A D) who has referred to many anterior writers, and (3) it contains the date of its last redaction, *viz*, 980 A V (=512 A.D)[4] Certain passages from one Bhadrabāhu have been quoted by Bhattotpala (966)[5]

Other Sources

We know of another Jaina astronomer of the name of Siddhasena, who has been referred to by Varāhamihira. Bhattotpala has quoted the corresponding passages from the works of Siddhasena. So it must have been existent at his time. It is lost now Besides, from the specific treatises on mathematics mentioned above, we can get lot of informations about the Jainas' knowledge of mathematics from the

[1] If the traditional date (527 B C) of the death of Mahāvīra be accepted, then the date of Bhadrabāhu's death will be 356 B C But we have here accepted the date suggested by Professor Jacobi on the authority of the great Jaina author Hemacandra (died 1172 A D), *viz*, 468 B C

[2] Sūtra 11, commentary

[3] *Report on Sanskrit Manuscripts* 1874 1875, p 20

[4] *Kalpasūtra* of Bhadrabāhu, edited by H Jacobi, Leipzig, 1897, Introduction, p 14

[5] *Brhat Samhitā* with the commentary of Bhattotpala, edited by Sudhakara Dvivedi, Benares, 1895, p 226

various Ardha Māgadhī religious and secular books For instance the commentator Śīlānka (862 A D) is found to have quoted three verses bearing on permutations and combinations which cannot be traced to any available treatise on mathematics [1] Elaborate specification of the dimensions of the different *dvīpas* or lands of the fantastic cosmography of the Jainas will certainly throw much light on the dark pages of the forgotten history.

Some valuable informations as regards the knowledge of mathematics amongst the early Jainas are expected to be found in the two other classes of works, *viz*, *Kṣetrasamāsa* and *Karaṇabhāvanā* or *Karaṇagāthā* There are several works of the name of *Kṣetrasamāsa* ("Collection of places"), the earliest of which is by Umāsvāti (*c* 150 B C) This last work is also known as *Jambudvīpasamāsa* Jinabhadra Gani (*c* 550 A. D) wrote two works of the same class · a bigger one, called *Bṛhat Kṣetrasamāsa* and a smaller one, called *Laghu Kṣetrasamāsa* Such treatises were composed by the Jaina scholars of the 13th and 15th centuries even They are expected to furnish informations as regards the knowledge of geometry amongst the Jainas. The other class of works *Karaṇabhāvanā* are believed to be older than the *Kṣetrasamāsa* They give in a nutshell the mathematical calculations employed in the Jaina canonical works Though I have met with many quotations from them by the various Jaina commentators of later times, original works have not come to my hands as yet.

Topics of Mathematics

According to the *Sthānānga sūtra*,[2] a Jaina canonical work of 300 B C or still earlier, the topics for discussion in mathematics (*saṁkhyāna* or the "Science of numbers") are *ten* in number *parikarma* ("fundamental operations"), *vyavahāra* ("subjects of treatment"), *rajju* ("rope," meaning "geometry"), *rāśi* ("heap," meaning "mensuration of solid bodies"), *kalāsavarṇa* ("fractions"), *yāvat-tāvat* ("as many as," meaning "simple equations"), *varga* ("square," meaning "quadratic equations"), *ghana* ("cube," meaning

[1] *Vide infra*, p. 134

[2] Sūtra, 747

परिकम्म ववहारो रज्जु रासी कलासवन्ने य ।
जावतावति वग्गो घनो त तह वग्गवग्गी विकप्पो त ॥

"cubic equations"), *varga-varga* ("biquadratic equations") and *vikalpa* ('permutations and combinations").

In explaining the above technical terms, specially those relating to algebra, we have departed so much from the opinion of the commentator Abhayadeva Sūri (1050 A D), that a justificatory explanation is necessary. The commentator has displayed much ignorance about the science of algebra Now the subjects of *parikarma, vyavahāra* and *kalāsavarṇa* will be readily recognised as they appear in the same form in the *Gaṇita-sāra-saṃgraha* of Mahāvīra, the only Jaina mathematician of later times whose works are available to us The first two terms appear indeed in the works of all Hindu mathematicians from Brahmagupta (628) onwards Though the term *rajju* does not appear in any later work, there will be no difficulty in recognising it as referring to plane geometry and as equivalent to the term *kṣetra* of later works It is synonymous with the term *śulba* of the Vedic period Hence *Rajju-saṃkhyāna* is identical with *Śulba-sūtra* [1] The commentator has rightly identified it with *Kṣetragaṇita*, a name for geometry appearing in the *Gaṇita-sāra-saṃgraha* [2] The term *rāśi* appears in later works, except this last mentioned one, and means measurements of maunds of grain. But

This verse has been quoted by Śīlāṅka (862 A D) in his commentary of *Sūtrakṛtāṅgasūtra* (2nd Śrutaskanda, ch 1, Sūtra 154) in a slightly modified form

परिकम्म रज्जु रासी ववहारे तह कलासवन्ने य ।
पुद्गल जावताव घने य घनवग्ग वग्गे च ॥

The editor of this last mentioned work has translated it into Sanskrit thus

परिकर्म्म रज्ज : राशि: व्यवहारस्तथा कलासवर्णश्च ।
पुद्गला: यावत्तावत् भवन्ति घन घनमूलं वर्ग: वर्गमूल ॥

This shows clearly that the editor has failed miserably in grasping the true sense of the second line of the original verse or of the modified one There is nothing in the either from which could be inferred a reference to "roots" (*mūla*) Above all by that interpretation, he has made the number of topics for discussion to be eleven, against the express injunction of the canonical work that they are altogether ten So we shall reject his reading of the verse For similar reasons we shall discard the modification of the commentator Śīlāṅka *Pudgala* as a topic for discussion in mathematics is meaningless

[1] Compare *Kātyāyana-Śulba pariśiṣta* (1 1) where Geometry is called *rajju-samāsa* "रज्जु समासं वक्ष्याम"

[2] Ch. vi

I do not think that it has been used in the same sense in the canonical works For measurement of heaps of grain has never been given any prominence in later mathematical works and indeed it does not deserve any prominence So it can be hardly believed that it was considered in the canonical works to be of such importance as to be counted as forming a separate section of the science of mathematics I am of opinion that *rāśi* means "heap" in general and hence refers to the section devoted to the treatment of the mensuration of solid bodies In the later Hindu treatises on mathematics, this section is named *khāta*, and the *rāśi* covers a very small portion of it

Hitherto we have practically no difficulty in interpreting some of the names given in the *Sthānāṅgasūtra* to the different sections of mathematics and in identifying them with the names given to the corresponding sections of the later mathematical treatises In the identification of the remaining terms, the commentator is not only of no help but is, on the other hand, positively misleading The culture of mathematics had deteriorated so much amongst the Jaina scholars of later times, inspite of the strict injunction of their religion to study mathematics, that they could hardly understand and appreciate the early scientific works In such circumstances it is not strange to find that they would make colossal blunders in explaining portions of mathematics, especially its analytical branch which requires keen and subtle intellect for proper understanding.[1] Abhayadeva surely thinks that *varga, ghana* and *varga-varga* refer respectively to the rules for finding the square, cube and fourth power of a number But in Hindu mathematics from the earliest times squaring and cubing are considered as fundamental operations and as such they are covered by the term *parikarma*. The method of finding the fourth power of a number has never been given a separate treatment in any work, for it is after all a case of squaring If it, however, be supposed for a moment that such consideration was probably given to it in the olden days of the canonical works, we ought

[1] In fairness to the commentator Abhayadeva Suri, it should be stated that he seems to have been acquainted with the arithmetical treatise of Śrīdhara, but not with his treatise on algebra For he has quoted portions of certain verses, e g, *saddrsadvirāśighātaḥ, samatrirāśehatiḥ* which can be traced to *Triśatikā* (Rules 11 and 15) But in attempting to explain *yāvat-tāvat*, he has quoted a peice of verse written in Prākṛta, and another verse containing an obscure mathematical principle which cannot be traced to any known work.

to have found separate sections for still higher powers of numbers, which, it will be shown later on, were known to the early Jainas. In such case, it will be more natural to expect to find a separate section for the method of finding the square root in which the Jainas were quite at hand. I have no doubt in my mind that *varga* refers to "quadratic equations," *ghana* to "cubic equations" and *varga-varga* to "biquadratic equations."

Abhayadeva Sūri held that *yāvat-tāvat* refers to multiplication or to the summation of series (*saṃkalita*). Now multiplication is included in the fundamental operations. And in referring to the alternative he has contradicted himself. For he has stated a little earlier that this latter subject is included in the section *vyavahāra* (*'vyavahāra' śreṇīh vyavahārādi*). This interpretation of the commentator can be objected also on another ground. In an explanatory note to his interpretation of *yāvat-tāvat*, he has quoted a rule for finding the sum (S) of n natural numbers together with an example which works out

$$S=\frac{n(nx+x)}{2x}$$

where x is an arbitrary quantity (*yaddrcchā*, *vāñchā* or *yāvat-tāvat*). Obviously the introduction of x is quite useless. I venture to presume that the term *yāvat-tāvat* is connected with the Rule of False Position which, in the early stage of the science of algebra in every country, was the only method of solving linear equations. It is interesting to find that this method was once given so much importance in Hindu Algebra, that the section dealing with it was named after it. The commentator, inspite of his other errors, is of opinion that *yāvat-tāvat* originated from *yaddrcchā*, meaning "an arbitrary quantity" or from *vāñchā* meaning "desired quantity." We find in the Bakhshālī mathematics, that both of these latter terms have been employed there in connexion with the rule of false position.[1] This Bakhshālī work was written about the beginning of the Christian era, and hence in a period not very far from the date on the canonical work. This will point to the correctness of our interpretation of the term *yāvat-tāvat* in the *Sthānānga-sūtra*.

[1] Bibhutibhusan Datta, "The Bakhshālī Mathematics," *Bull. Cal. Math. Soc.* Vol XXI, 1929, No. 1, pp. 1-60.

The commentator has acted most foolishly in explaining the latter portion of the verse. He thinks that *vaggavaggo vikoppo ta* should be analysed as *varga-vargah api kalpah tathā* and says that the section on *kalpa* deals with what is called "saw" in later works. Obviously the construction of this portion of the verse should be *vargavargah vikalpah tathā*. It will be shown later on that the early Jainas attached great importance to the subject of permutations and combinations (*vikalpa*). So it will be quite natural that a section of their treatises on mathematics should be devoted to its treatment.

One term in the list of topics of mathematics as stated above deserves particular notice. It is the term *yāvat-tāvat*. That term enters largely into Hindu Algebra of later times as the symbol for the unknown. It has been suggested that it is connected with the definition of the unknown quantity given by the Greek Diophantus (c. 75 A. D.) as "containing an indeterminate or undefined multitudes of units" (*pléthos monádon aoriston*).[1] The implication behind that suggestion was to show the Greek influence in the Hindu Algebra. It is now found that *yāvat-tāvat* has entered into Hindu mathematics more than five centuries before Diophantus. So if that suggestion be at all true, though I doubt it, it will have to be admitted that the balance of evidence is in favour of the Hindus, showing the possibility of the Greek Algebra being influenced by the Hindu Science. This will take aback the author of that suggestion.

The ancient work *Cūrṇi* defines the terms *parikarma* as referring to those fundamental operations of mathematics as will befit a student to enter into the rest and the real portion of the science. According to it the fundamental operations are sixteen in number.[2] It may be pointed out that Brahmagupta makes the number twenty and all others have reduced it to eight.

Sthānāṅga-sūtra[3] considers mathematics (*gaṇita*) including permutations and combinations (*bhaṅga*) to be very subtle (*sukṣma*). The commentator observes that those subjects are considered subtle as their study requires subtle intellect. He further adds that though permutations and combinations are really included into mathematics, they have been accorded a separate mention on account of their

[1] G. R. Kaye, *Indian Mathematics*, Calcutta, 1915, p. 25.

[2] Quoted in the Jaina Encyclopaedia, *Abhidhāna Rājendra*.

[3] Sūtra 716.

importance.[1] This canonical work has once referred to the "elements of mathematics" (*ganitasya ca bījānām*).[2] The author probably meant thereby the science of algebra (*bījaganita*). For we have seen before that he included topics of algebra in enumerating the topics of mathematics. In the opinion of the *Sūtrakṛtanga-sūtra*,[3] "geometry is the lotus in mathematics, and the rest is inferior."

Certain Mensuration formulæ.

In the *Tattvārthādhigama-sūtra bhāṣya*[4] of Umāsvāti is found the incidental reference to the following mensuration formulæ. If C denote the circumference of a circle of diameter d and area A, then

$$(1)\quad C=\sqrt{10d^2},$$

$$(2)\quad A=\frac{1}{4}\,Cd$$

Again if a denotes the arc of a segment of the circle less than a semicircle, c its chord and h its height or arrow, then

$$(3)\quad c=\sqrt{4h(d-h)},$$

$$(4)\quad h=\frac{1}{2}\left(d-\sqrt{d^2-c^2}\right),$$

$$(5)\quad a=\sqrt{6h^2+c^2}.$$

$$(6)\quad d=\left(h^2+\frac{c^2}{4}\right)/h$$

[1] This will support our paraphrase of the latter portion of the verse of the Sūtra 747, that *vikalpa* refers to the section on permutations and combinations.

[2] *Sthānanga sūtra*, Sūtra 673.

[3] 2nd Śrutaskanda, ch. 1, verse 154.

[4] *Tattvārthādhigama-sūtra* with the *Bhāṣya* of Umāsvāti, edited by K. P. Mody, Calcutta, 1903. An excellent edition of this work together with the notes of Siddhasena Gaṇi is in course of publication by Professor H. R. Kapadia of Bombay. The Part I, containing the chapters I-V., is already out.

(7) The portions of the circumference of the circle between two parallel chords is half the difference between the corresponding arcs.

All these formulæ, with the exception of (4), are restated in the *Jambudvīpasamāsa*[1] of Umāsvāti In this work, the method of finding the arrow is stated thus

$$(8)\ h=\sqrt{(a^2-c^2)/6}$$

Multiplication and Division by factors

In the *Tattvārthādhigama-sūtra-bhāṣya* [2] of Umāsvāti, there is also an incidental reference to two methods of multiplication and division. One is our ordinary method, in which the respective operations are carried on with the two numbers considered as whole According to the other method, the operations are carried on in successive stages by the factors, one after another, of the multiplier and the divisor It has been found that the final result obtained by the either methods is the same but that the second method is shorter and simpler than the other The multiplication by factors has been mentioned by all the known Hindu mathematicians from Brahmagupta [3] (628) onward The division by factors is found in the *Triśatikā* [4] of

"विष्कम्भकृतेर्दशगुणाया मूलं वृत्तपरिक्षेपः। स विष्कम्भपादाभ्यस्तो गणितम्। इच्छावगाहोनावगाहाभ्यस्तस्य विष्कम्भस्य चतुर्गुणस्य मूलं ज्या। ज्याविष्कम्भयोर्वर्गविशेषमूलं विष्कम्भाच्छोध्यं शेषार्धमिषुः। इषुवर्गस्य षड्गुणस्य ज्यावर्गयुतस्य मूलं धनुःकाष्ठम्। ज्यावर्गचतुर्भागयुक्तमिषुवर्गमिषुविभक्तं तत्प्रकृतिवृत्तविष्कम्भः। उदग्धनुःकाष्ठाद्दक्षिणं शोध्यं शेषार्धं वाहुरिति। अनेन करणाभ्युपायेन सर्व्वक्षेत्राणां सर्व्वपर्व्वतानामायामविष्कम्भज्येषुधनुःकाष्ठपरिमाणानि ज्ञातव्यानि।"

Ch iii, sūtra 11 (Bhāṣya)

[1] This work has been published in the Appendix C of Mody's edition of the *Tattvārthādhigamasūtra* noted above

"विष्कम्भवर्गदशगुणकरणीवृत्तक्षेत्रपरिधिः। विष्कम्भपादाभ्यस्त स गणितम्। विष्कम्भोऽवगाहोनस्तद्गुणचतुर्गुणष्टन्मूलं जया। इषुवर्गः षड्गुणो जयावर्गं क्षिप्तस्तन्मूलं धनुःपृष्ठम्। चतुर्गुणेषु युक्तविभक्तो जयावर्गो विष्कम्भः। धनुर्वर्गजयावर्गविशेषषड्भागमूलमिषु। दक्षधनुःपृष्ठापनीतह्रस्वधनुःपृष्ठार्धं वाहु।" *Âhnik* 4,

[2] ii 52

[3] *Brāhma sphuta siddhānta*, xii 55. Brahmagupta calls it the *Bheda* method, others call it *Vibhāga gunana* Compare H T Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, London, 1817, p 6 fn.; hereafter referred to as Colebrooke, *Hindu Algebra*

[4] Rule 9

Śrīdhara (*c.* 750). They went to Italy in the middle ages, through Arabia, and were called there the "modo per repiego." [1]

Umāsvāti

Though Umāsvāti is reputed to be one of the greatest metaphysicians of India and though he is held in high estimation equally by the two main sections of the Jainas, it is unfortunate that neither the time nor the place of his birth has been settled definitely up to this time According to the tradition of the Śvetāmbara Jainas, Umāsvāti was born in the now forgotten city of Nyagrodhikā His name is said to have been a combination of the names of his parents, the father Svāti and the mother Umā He was the disciple of the saint Ghosanandi. He lived about 150 B.C His disciple Śyâmârya or Śyâmâcârya, the author of the *Prajñapanā-sūtra* is said to have died 376 years after Śrī Vīra, that is, in 92 B C. and his earliest commentator is said to have been Siddhasena Gaṇi, or Divākara who lived *c.* 56 B C The Digambara tradition, on the other hand, sometimes even changes his name and thinks it to be Umāsvāmī, not Umāsvāti According to it he lived in the years 135 A.D.-219 A.D Satischandra Vidyabhusan is of opinion that he flourished in the first century A D. All are, however, agreed on one point, that Umāsvāti resided in the city of Kusumapura (ancient Pāṭalīputra, near modern Patna) [2]

The Kusumapura School of Mathematics

It is noteworthy that Umāsvāti's name has come down to us as a great writer on the Jaina doctrines, but not as a writer on mathematics. He is not even known to have ever devoted himself to a study of this science Hence it will have to be concluded that the mathematical formulæ quoted in his *Tattvārthādhigama-sūtra-bhāṣya* were taken from some other treatise on mathematics known at his

[1] D E Smith, *History of Mathematics*, in two volumes, Boston, Vol II, pp 101, 135, hereafter referred to as Smith, *History*

[2] *Vide* the preface to Kapadia's edition to *Tattvārthādhigama sūtra* and Peterson's *Fourth Report of operation in search of Sanskrit Mss in the Bombay Circle*, 1886 1892, pp xvi-xvii,

time.[1] The method of multiplication and division by factors must have been very familiar to the intelligentia of his time. Otherwise, Umāsvāti would not have taken recourse to it as a metaphor to establish certain category of his philosophical speculations. Thus we come to learn of the existence of a school of mathematics at Kusumapura, near about the beginning of the Christian era. It must have come into being long before. For it will be remembered that the famous Jaina saint Bhadrabāhu (c. 150 A.V. or 318 B.C.) lived at Kusumapura and was the author of two astronomical works, a commentary on the *Sūryaprajñapti* and the *Bhadrabāhavī Saṃhitā*. The culture of mathematics and astronomy survived in this school up to the end of the fifth century of the Christian era when flourished the famous Āryabhaṭa (born 476 A.D.) who is reputed for his many innovations in the Hindu astronomy and who has been almost unanimously acknowledged by the later mathematicians as the father of the Hindu Algebra. There is evidence to show that the influence of this school of mathematics continued unabated for several centuries after Āryabhaṭa.[2]

Its relation with other Schools of Mathematics

Two other important and well known centres of mathematical culture in ancient India were Ujjain and Mysore. The Ujjain School included Brahmagupta and Bhāskarācārya, the greatest of Indian astronomers and mathematicians, while the Southern School of Mysore had its representative in Mahāvīrācārya. It will be interesting to know what were the relations of these schools with the Kusumpura School of mathematics. About 155 A.V. (=313 A.D.), a terrible famine is said to have devasted the realm of Magadha. It lasted for 12 years. In that terrible time one section of the Jaina community of Magadha, headed by their priest Bhadrabāhu emigrated to Southern

[1] Such has also been the opinion of the commentator Siddhasena: "अपरे पुनार्वैद्दासोऽतिवहुनि स्वय विरचय्यास्मिन् प्रस्तावे सूत्राग्यवोयते विस्तरदर्शनाभिप्रायेग, तत्स्वयुक्तमयं संग्रह: सुरिणा संचेप: क्रत इत्यतोऽच विस्तराभिधानमप्राचीनमाचचते प्रवचननिपुणा:" (iii, 11),

[2] There are strong reasons to believe that there was another astronomer and mathematician of the name of Āryabhaṭa at Kusumapura who was anterior to the Āryabhaṭa of 476 A.D. We hear of also many followers of this latter Āryabhaṭa, some amongst whom rose to eminence. Compare "Two Āryabhaṭas of Al-Birûnî," *Bull. Cal. Math. Soc.*, Vol. 17, 1926, p. 68.

India and settled near Sravana Belgola in Mysore On his way he passed through Ujjain and halted there for sometime This tradition is supported by local tradition, several inscriptions and literature. The earliest of those inscriptions is dated 650 A D We have already stated that Bhadrabāhu was not only an eminent religious teacher, but also an astronomer and a mathematician Thus connexion between the three importants schools of Hindu mathematics is learnt to have been established in very early times But in the absence of specific records, we are not in a position to give any further idea about the character and extent of their mutual relation. We cannot say if Mahāvīra's obligation to an early Jaina mathematician, who is described as *Jinendra*, or " The Great Jina " has any reference to Bhadrabāhu This scholar priest fully deserves that epithet

Discovery of the mensuration formulæ

It has been observed before that Umāsvāti is not probably the discoverer of the mensuration formulæ that are now found recorded in his works. In fact, there are reasons to prove that the most of those formulæ were known centuries before him. In the *Sūrya-prajñapti* (c 500 B C)[1] and other early Jaina sūtras are stated the length of the diameter and circumference of certain circular bodies (*vide supra*) These results are in accord with the formulæ stated before According to the Jaina cosmography, the Jambudvīpa which is circular with a diameter of 100,000 *yojana*, is divided into seven parts by a system of six mountain ranges running parallel, east to west, at regular intervals The *Jambudvīpa-prajñapti* (c 500 B C) gives the linear dimensions of each of these parts [2] For instance we quote the dimensions of the Bhāratavarsa, which forms the southernmost segment of the Jambudvīpa [3] Its breadth, (i e, the height of the circular segment) is $526\frac{6}{19}$ *yojana*, its length (i e, the chord of the segment) is a little over $14471\frac{6}{19}$ *yojana*, and the length of its southern

[1] Sūtra 20

[2] W Kirfel, *Die Kosmographie der Inder*, Bonn, 1920, p. 216.

[3] *Jambudvīpaprajñapti* with the commentary of Sānticandra Gaṇi, edited by Agamodayasamiti of Mehasana, 1918, Sūtra, 10 12, 16.

boundary (i e , the arc of the segment) is $14528\frac{11}{19}$ *yojana* A mountain, called Vaitāḍhya, of the depth of 50 *yojana*, is said to be running through the middle of the Bhāratavarṣa parallel to its length The northern and southern sides of this mountain are given as $10720\frac{12}{19}$ and $9748\frac{12}{19}$ *yojana* respectively Further the portions of the bounding arc cut off by the two parallel sides are given to be $488\frac{16}{19}+\frac{1}{38}$ *yojana* each All these numerical calculations prove conclusively that most of the mensuration formulæ recorded by Umāsvāti were well-known to the author of the *Jambudvīpaprajñapti* They also occur in the ancient work *Karanabhāvanā*

In the *Uttarādhyayana sūtra* (c. 300 B C), we find the following description of Īsatprāgbhāra, "which resembles in form an open Umbrella," i e , the segment of a sphere "It is forty-five hundred thousand yojanas long, and as many broad, and it is somewhat more than three times as many in circumference Its thickness is eight yojanas, it is greatest in the middle,and decreases towards the margin, till it is thinner than the wing of a fly"[1] The *Aupapātika-sūtra* further specifies the circumference to be 14239800 *yojana* and it is also said that the depth decreases an *angula* for every *yojana*.[2] This description strongly suggests a knowledge of mensuration of a spherical segment amongst the early Jainas

It may be noted here that the formula for the arc of a segment less than a semicircle reappears in the *Ganita-sāra-samgraha*[3] of Mahāvīra (850) and the *Mahāsiddhānta*[4] of Āryabhata II (950) According to the former

$$a \text{ (gross)} = \sqrt{5h^2+c^2}$$

$$a \text{ (neat)} = \sqrt{6h^2+c^2}$$

1 *Uttarādhyayana-sūtra*, XXXVI 59-60

2 *Aupapātikasūtra*, ed Leumann, § 163-7

3 *Ganita sāra-samgraha*, VII 43, 73½

4 *Mahāsidhānta* of Āryabhata II, edited by Sudhakara Dvivedi, Benares, 1910, XV 90, 91 95

and according to the latter,

$$a \text{ (gross)} = \sqrt{6h^2 + c^2}$$

$$a \text{ (neat)} = \sqrt{\frac{288}{49} h^2 + c^2}$$

The Greek Heron of Alexandria (c 200) takes the circumference of the segment less than a semicircle to be[1]

$$\sqrt{4h^2+c^2} + \tfrac{1}{4}h$$

or

$$\sqrt{4h^2+c^2} + \left\{ \sqrt{4h^2+c^2} - c \right\} \frac{h}{c}$$

The Chinese Ch'ên Huo (died 1075) gives the formula[2]

$$a = c + 2\frac{h^2}{d}$$

It will be observed that the Hindu value of the arc is older and more accurate than the other two It should be further noted that the formula (4) requires the solution of a quadratic equation We do not find amongst the Hindus, as far as is known, any expression for the area of a segment of a circle before the time of Śrīdhara[3] (c 750) though it was known in Greece and China long before.[4]

[1] T Heath, *History of Greek Mathematics* in two volumes, Oxford, 1921, Vol II, p 331 Hereafter this book will be referred to as Heath, *Greek Mathematics*

[2] Y Mikami, *The Development of Mathematics in China and Japan*, Leipzig, 1913, p 62, hereafter referred to as Mikami, *Chinese Mathematics*

[3] *Triśatikā* of Śrīdhara, edited by Sudhakara Dvivedi, Benares, 1899, Rule 47. This formula has been quoted by Ganeśa (1545) in his commentary of Bhāskara's *Līlāvatī*. Compare also *Ganita sāra-samgraha*, VII 43, 78½, *Mahāsidhānta*, XV 89, 93, 94

[4] Heath *Greek Mathematics*, II, p 330, Mikami, *Chinese Mathematics*, pp. 11, 22, 39

Jaina value of π $\left(=\sqrt{10}\right)$

In the *Sūryaprajñapti*[1] (c 500 B C.) we find reference to two values of π. One is $\pi=3$ and the other is $\pi=\sqrt{10}$ The former is due to earlier writers and has been discarded by the author The latter value of π has been approved by him and adopted throughout the early Jaina literature [2] And it continued to be so even in the Jaina works written as late as in the fifteenth century when the Hindus had discovered more accurate values [3] Hence Professor Mikami is not correct in stating that the value $\pi=\sqrt{10}$ is found recorded in a Chinese work before it appeared in any Hindu work [4] For Chang Heng who is said to have recorded this value first among the Chinese lived in the years 78-139 A.D [5] whereas the *Sūryaprajñapti* is referred to c 500 B.C In the *Uttarādhyayana-sūtra*[6] (before 300 B.C.), the circumference is stated roughly to be a little over three times its diameter It is stated in the *Jīvābhigamasūtra*[7] that for an increment of 100 in the diameter, the circumference increases by 316 This gives $\pi=3\ 16$

[1] Sūtra 20

After referring to the dimensions of the solar orbit according to three older schools—all of which work out $\pi=3$—Mahāvīra says that according to him the diameter of the innermost orbit of the sun is 99640 *yojana* and its circumference is 315089 *yojana* and a little over (*kiñcidviśeṣādhika*) He then states sets of other values for the dimension of the successive orbits $d=99645\frac{35}{61}$, $C=315107$ and a little less (*kiñcidviśeṣuna*), $d=99651\frac{9}{61}$, $C=315125$, $d=100660$, $C=318315$ (*kiñcidviśeṣuna*), etc All these are clearly based on the relation

$$C=\sqrt{10d^2}$$

Sūryaprajñapti contains also other instances of the application of this formula For an explanation about the origin of the value $\pi=\sqrt{10}$ see the author's paper on the "Hindu values of π" (*Journ Asiat Soc. Beng*, Vol 22, 1926)

[2] For instance, *Jīvābhigama-sūtra*, Sūtra 82, 109, 112, etc *Jambudvīpa-prajñapti*, Sūtra 3, *Bhagabatī-sūtra*, Sūtra 91, *Tattvārthādhigama-sūtra-bhāṣya*, iii 11.

[3] See *Laghukṣetrasamāsaprakarana* of Ratnaśekharasūri (1440 A D) included in the *Prakarana Ratnākara* edited by Bhīmasimha Mānaka, Bombay, 1881, verse 187

[4] Mikami, *Chinese Mathematics*, p 70

[5] *Ibid*, p 46

[6] XXXVI 59 Compare also *Jambudvīpaprajñapti* (Sūtra 19)—*trigunam saviśesam* (a little over three times)

[7] *Jīvābhigama sūtra*, Sūtra 112.

Approximate values of surds.

In the Jaina sacred books,[1] (c 500 B C) the dimensions of the Jambudvīpa which is circular in shape, are given as follows —diameter = 100000 *yojana*, circumference = 316227 *yojana* 3 *gavyuti* 128 *dhanu* 13½ *angula* and a little over, and area = 7905694150 *yojana* 1 *gavyuti* 1515 *dhanu* 60 *angula* nearly It will be easily seen that in calculating these values of circumference and area from the given value of the diameter, using $\pi=\sqrt{10}$ and the formulæ $C=\sqrt{10d^2}$ and $A=\frac{1}{4}Cd$, there has been followed a principle of approximation to the value of a surd which may be expressed as

$$\sqrt{N}=\sqrt{a^2+\epsilon}=a+\frac{\epsilon}{2a}$$

Modern historians of mathematics erroneously attribute this approximate square-root formula to Heron of Alexandra (c 200 A.D),[2] but the credit for its first discovery is truly due to the Hindus.

Approximate values of big fractions

In the Jaina works we notice another kind of approximation In a mixed number if the fractional part is greater than ½, it is replaced by 1, on the other hand if it be less than ½ it is neglected So that for practical purpose the value of a quantity is often times stated in round numbers with the observation that the two value of the quantity is either a little more (*kiñcidviśesādhika*) or a little less k(*iñcidviśesūna*). For instance,[3] the caculated value of the circumference of a circle whose diameter is 99640 *yojana* will be $315089\frac{218079}{630178}$ *yojana* according to the approximate square-root rule noted above This latter value is expressed in round numbers as " a little over " 315089.

[1] *Jambudvīpaprajñapti*, Sūtra 3, *Jīvābhigama sūtra*, Sūtra 82, 124, *Anuyogadvāra sūtra*, Sūtra 146

It is noteworthy that this relation between the diameter of a circle and its cicumference has been stated in a general way in the *Jīvābhigamasūtra* without any reference to the Jambudvīpa

[2] Smith, *History* II, p 254

[3] *Vide supra*, p 131, footnote 1.

Similarly the calculated value for the circumference, when the diameter is 100660, is $318314\frac{553404}{636628}$ and it is stated as "a little less than" 318315. Again for an increase or decrease in the diameter of a circle by $5\frac{35}{61}$, the change in the circumference ought to have been $17\frac{35}{61}$ but it is stated in round numbers to be 18.[1]

Permutations and Combinations

The early Jainas seem to have great liking for the subject of combinations and permutations (*bhanga* or *vikalpa-ganita*). For they are found to have employed their knowledge of that branch of mathematics in the various fields of their thought.[2] In the *Bhagabatī-sūtra* (c. 300 B.C.), we find instances of speculation about the different philosophical categories that can arise out of the combination of n fundamental categories taken one at a time (*ekakasamyoga*), two at a time (*dvika samyoga*), three at a time (*trikasamyoga*), or more at a time.[3] Similarly we have calculations of the groups that can be formed out of the different instruments of senses (*karaṇas*),[4] or of the selections that can be made out of a number of males, females and eunuchs,[5] or of combinations and permutations of various other things.[6] In all cases, they have succeeded to find the correct results

[1] The ancient work *Karanabhāvanā* remarks सत्तरस जोयणाइं अट्तीस च एग- ट्ठिभागा एयं निच्छतया सव्वहारेण पुन अट्ठारस जोयणाइं

$$5\frac{35}{61} \times \sqrt{10} = \sqrt{10 \times \frac{340 \times 340}{61 \times 61}} = \frac{1035}{61} = 17\frac{35}{61} \text{ approximately}$$

[2] Similar great interest for the subject of combinations and permutations was evinced by the early Hindu writers who applied it in the field of philosophy, medicine, astrology, and also other subjects.

[3] *Bhagabatī-sūtra* (Sūtra 314).

[4] *Ibid*, viii. 5.

[5] *Ibid*, viii. 8 (S. 341).

[6] *Ibid*, ix. 32 (S. 371. 4). Cf. *Jambudvīpa prajñapti*, xx. 4, 5, *Anuyogadvāra sūtra*, Sūtra 76, 92, 126.

which will be now expressed as

$$ {}^nC_1 = n, \quad {}^nC_2 = \frac{n(n-1)}{1 \cdot 2}, \quad {}^nC_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} $$

$$ {}^nP_1 = n, \quad {}^nP_2 = n(n-1), \quad {}^nP_3 = n(n-1)(n-2) $$

After having stated the results for $n=1, 2, 3, 4$, the author observes "And in this way five, six, seven, ten, etc., numerable, innumerable or infinite number of things may be mentioned. Taking one at a time, two at a time, three at a time, , ten at a time, twelve at a time, as the number of combinations are formed, all of them must be considered."[1]

The Jaina commentator Śīlānka (862 A.D.) has quoted three rules regarding permutations and combinations.[2] Two of them are in Sanskrit verse and the other is, most interestingly, in Ardha Māgadhī verse. We do not know up till now of any treatise of mathematics written in Ardha Māgadhī. Nor can the two Sanskrit verses be traced to any known work. Here is then the most conclusive evidence of the existence of two early treatises on Mathematics which are now lost. The first rule is for determining the total number of transpositions that can be made with a specific number of things (*bheda-saṁkhyā-parijñanāya*):

> "Beginning with unity up to the number of terms, multiply continually the (natural) numbers. That should be known as the result in the calculation of permutations and combinations (*vikalpa-gaṇita*)."[3]

That is if n be the number of different things given, then the total number of permutations that can be made with them will be given by

[1] *Bhagabatî sūtra*, viii 1 (3 314) "एवम् एतेन क्रमेण पञ्चषट्सप्त यावत् दश संख्येयानि असख्येयानि अनन्तानि च द्रव्याणि भणितव्यानि। एककसयोगेन द्विकसयोगेन त्रिकसयोगेन यावत् दशसंयोगेन द्वादशसयोगेन उपयुज्य यथा यथा सयोगा उत्तिष्ठन्ति ते सर्व्वे भणितव्या · ।"

For the Sanskrit rendering of the Ardha Māgadhī original of this and other passages in this paper, I am indebted to Pandit Prabhat Kumar Mukerjee, Research Scholar, Calcutta University.

[2] Vide his commentary on the *Sūtrakṛtānga-sūtra, samayādhyayana, anuyoga-dvāra*, verse 28.

[3] "एकाद्या गच्छपर्य्यन्ताः परस्परसमाहताः।
राशयस्तद्धि विज्ञेय विकल्पगणिते फलम्॥"

$1.2.3 \ldots\ldots\ldots (n-1)n$. The other rules are for finding the actual spread or representation (*prastāranayanopaya*). The Sanskrit verse may be rendered as follows

"The total number of permutations being divided by the last term, the quotient should be divided by the rest. They should be placed [illegible] initial term in the calculations of permutations and combinations."[1]

The rule appears to be cryptic, but Silanka has clearly explained it with the help of an illustrative example.[2] Let there be r number of things $a_1, a_2, \ldots a_r$. Then the total number of permutations that can be made with them will be by the previous rule $1.2.3. \ldots (r-1)r$ or $r!$ The number of permutations which can have any particular thing, say a_r, for its initial digit (*ādi*) will be [illegible], that is

1 " गणितेऽन्त्यविभक्ते तु लब्धं शेषैर्विभाजयेत् ।
आदावन्ते च तत स्थाप्य' विकल्पगणिते क्रमात् ॥ "

The Ardha Māgadhī verse is (*karaṇa gāthā*)

" पुव्वानुपुव्वि हेट्ठा समयाभेएण कुण जहाजेट्ठं ।
उवरिमतुल्लं पुरओ नसेज्ज पुव्वक्कमो सेसे ॥ "

2 Silanka says अयं श्लोकः [illegible] [illegible] समाश्रित्य तावत् श्लोकार्थो योज्यते । तत्रेदं १२३४५६ [illegible] [illegible] सप्तशतानि विंशत्युत्तराणि गणितमुच्यते । [illegible] शत लभ्यते । तत्र यथा [illegible] शतम् । [illegible] विंशत्युत्तराणि भवन्ति । एवा च गणितप्रक्रिया [illegible] पुनः शेषेण पञ्चकेन भागेऽपहृते लब्धा चतुर्विंशतिः । [illegible] पञ्चमपङ्क्तौ न्यस्याः यावद्विंशत्युत्तरं शतमिति । [illegible] संख्यः स सोऽधस्ताच्चतुर्विंशतिसंख्य एव तावत् [illegible] पूर्णानि भवन्ति । एवा च गणितप्रक्रिया [illegible] शेषचतुष्कं [illegible] भागे हृते षट् लभ्यन्ते, तावन्तश्चतुर्थपङ्क्तौ [illegible] भूय एककाः । पुनः पूर्वन्यायेन पङ्क्ति [illegible] तावन्मात्रो [illegible] तृतीयपङ्क्तौ । [illegible] व्यवस्थाप्यमिति ।

I hope to be excused for a lengthy quotation. As a systematic scheme of finding all the permutations with a given number of things is not found in any other Hindu treatises on mathematics I have thought the scheme quoted by Silanka worth recording. This scheme has also been noted by Hemacandra Sūri (born in [illegible] A.D.) in his commentary on *Anuyogadvara-sutra*, *Sutra* 97. The method of calculating the total number of permutations is indicated in the original *sutra* (*cf.* also Sutra 113, 115).

$(r-1)!$ So put a_r in the beginning of $(r-1)!$ number of grooves. Similarly put a_{r-1} in the beginning of another $(r-1)!$ grooves and so on Again amongst the first series of grooves, the number of sub-groves that can have a_{r-1} after a_r will be $(r-1)!/(r-1)$ or $(r-2)!$ Place a_{r-1} after a_r in those sub-grooves. The number of sub-grooves that can have a_{r-2} after a_r will be $(r-2)!$ and put it after a_r in those sub-grooves. Similarly with $a_{r-3}, a_{r-4}, \ldots a_1$ Again amongst the sub-grooves that can have any other particular thing in the third place will be $(r-2)!/(r-2)$ or $(r-3)!$ and it should be placed in those cases. Proceeding step by step in this way in a systematic manner we can find out all the different permutations of things

Law of indices

In the *Anuyogadvāra-sūtra,* a canonical work written before the commencement of the Christian era,[1] the total number of human beings in the world is given thus "a number (which when expressed) in terms of the denominations *koti-koti*, etc., occupies twenty-nine places (*sthāna*), or it is beyond the twenty-fourth place (*pada*) and within the thirty-second place,[2] or a number obtained by multiplying sixth square by the fifth square, or a number which can be divided by two) ninety-six times"[3] It is indeed a very large number and the different specifications seem to have been necessitated to indicate it correctly.

We shall first indicate how the higher powers of a number are defined in the work Of course we do not find powers other than the

[1] J Charpentier, *Uttarādhyayana sūtra*, Upsala, 1922, Introduction, pp 29 30

[2] The text has · *triyāmalapadasya upari caturyamalapadasya adhastāt.* According to the celebrated Jaina writer Hemacandra (born 1089 A D), who has written a commentary on the *Anuyogadvāra-sūtra*, *yamala-pada* is a technical term which admits of two interpretations, *either* (1) A group of eight notational places form a *yamala pada* Hence the number defining the human beings lies above the 24th place and below the 32nd place Or (2) *triyamala pada* means "the sixth square' and *caturyamala pada* means "the eighth square," so that the number in contemplation is stated to be lying between the sixth square and the eighth square. The either interpretation will fit in

[3] *Anuyogadvāra sūtra*, Sūtra 142

successive squares (*varga*) and square-roots (*varga-mūla*)[1]

1st square of a means		$(a)^2 = a^2$
2nd ,, ,, ,, ,,		$(a^2)^2 = a^4$
3rd ,, ,, ,, ,,		$(a^4)^2 = a^8$
...	...	...
nth square of a means		a^{2^n}

Similarly,

1st square-root of a means		$\sqrt{a} = a^{\frac{1}{2}}$
2nd ,, ,, ,, ,,		$\sqrt{(\sqrt{a})} = a^{\frac{1}{4}}$
3rd ,, ,, ,, ,,		$\sqrt{(\sqrt{(\sqrt{a})}} = a^{\frac{1}{8}}$
...	...	...
nth square-root of a means		$a^{1/2^n}$

It will be noticed here, as in the later Hindu treatises on algebra, that powers of a quantity have been indicated on the multiplicative principle. So that in general

nth *varga* of a means $a^{2\times 2\times 2\times}$ to n terms $= a^{2^n}$

Similarly we have for the *vargamūla* Again we find that

3rd *vargamūla-ghana* of a means $(a^{1/2^3})^3 = a^{3/8}$

The mode of indicating powers of a quantity is more clearly stated in the *Uttarādhyayana-sūtra* (c 300 B.C or earlier).[2] According to it the second power is called *varga* ("square"), the third power *ghana* ("cube"), the fourth power *varga-varga* ("square-square"), the sixth power *ghana-varga* ("cube-square") and the twelfth power is called *ghana-varga-varga*. These terms reappear in all the Hindu treatises on mathematics It should be noted that in this word we do not find any method for indicating the odd powers, such as the fifth, seventh, etc.

[1] It has been stated by the writer in a previous paper that the use of the word *mūla* in the theory of numbers in the sense of "root" occurs in the *Āryabhaṭīya* (499 A.D) ("On *Mūla*, the Hindu term for 'root'," *Amer Math Monthly*, XXXIV, 1927, pp. 420-423) It is now found that the same use began before the beginning of the Christian era. It should be further noted that in this work the term *mūla* has a *concrete* as well as an *abstract* concept

[2] XXX. 10, 11.

Now the number of human beings (N) will be obtained by multiplying the sixth power of 2 by its fifth power [1] So that

$N = 2^{64} \times 2^{32} = 2^{96}$

$= 79,\ 228,\ 162,\ 514,\ 264,\ 337,\ 593,\ 543,\ 950,\ 336$

This is a very large number indeed and it actually occupies 29 notational places, as stated in the work. And obviously the number can be halved 96 times.

We also meet with in the *Anuyogadvāra-sūtra*[2] such statements as "first square-root, multiplied by the second square-root, or the cube of the second square-root, and the second square-root multiplied by the third square-root, or the cube of the third square-root" All these come in connexion with certain calculations in which each result has been specified by two alternative ways.[3] Expressed symbolically, they will be

$$a^{\frac{1}{2}} \times a^{\frac{1}{4}} = (a^{\frac{1}{4}})^3$$

and
$$a^{\frac{1}{4}} \times a^{\frac{1}{8}} = (a^{\frac{1}{8}})^3$$

After what has been stated above there will remain little doubt that the early Jainas knew the law of indices

$$a^m \times a^n = a^{m+n}, \qquad (a^m)^n = a^{mn}$$

where m, n may be integral or fractional.

Place-value system of notation.

Another very interesting and noteworthy point in the passage quoted above is that it contains reference to the "places" (*sthāna*) of decimal numerical notation the denominationl names *koṭi-koṭi*, etc. are indicated to be referring to the "places of numerations" There

[1] Why the base number is usually taken to be 2 in this case, though it is not expressly indicated in the work, we cannot say

[2] Sūtra 142

[3] It is noteworthy that these statements are of so general character that it will not be right to say that they are only arithmetical They indicate rather generalised arithmetic or algebra

is further mention of a very large number extending over "twenty-nine places." The reference to the "places" of calculation (*gaṇanā-sthāna*) also occurs in the *Vyavahāra-sūtra*.[1] All these will strongly lead to the conclusion that the place-value system of decimal notation was known in India in the centuries earlier than the commencement of the Christian era.

We cannot say what were the forms of the numerals used by the early Jainas. That they had some numeral characters, we have no doubt. For as early as the fourth or fifth century before the Christian era we find in a list enumerating the different written characters (*lipi*) known about that time, the mention of *aṅkalipi* and *gaṇita-lipi*.[2] That list has been reproduced in the *Prajñāpanā-sūtra*[3] of Syâmâiya who died in 376 A. V. (=92 or 151 B.C.). These two names suggest further that the forms of numerals used for different purposes were different. The former refers to the numerals used in engraving and the latter to those used in ordinary writing. In the Jaina literature, as also in the Vedic literature we ordinarily find that a distinction is made beeween forms of alphabets used in engravings, (called by the Jainas *kāsṭakarma* or "wood-work") and in manuscripts, (called *pustaka-karma* or "book-work").[4] This reference is very important inasmuch as it shows how one-sideed and partial are the views of those writers who consider the origin and development of the Hindu numerals on the palæographic evidence only.

It may be noted that the numerical vocabulary found ordinarily in the early Jaina literature is in certain respects different from that found in the Vedic literature. Whereas in the latter there are distinct and special names for each of the units of different denominations, in the former on the other hand, the necessary terminologies, above the fourth denomination, have been coined by a cumbrous system of grouping and regrouping.[5] Thus we have the following numerical vocabulary: units (*eka*), tens (*daśa*), hundreds (*śata*), thousands (*sahasra*), tens of thousands, hundreds of thousands,

[1] Ch. 1.

[2] *Samavāyāṅga-sūtra*, Sūtra 18.

[3] *Prajñāpanā-sūtra*, Sūtra 37.

[4] *Anuyogadvāra-sūtra*, Sūtra 146, compare also Sūtra 10 and its commentary.

[5] Compare Bibhutibhusan Datta, "The present mode of expressing numbers," *Ind. Hist. Quart.*, III, 1927, pp. 530-540.

tens of hundreds of thousands, *koṭi*, tens of *koṭi*, hundreds of *koṭi*, etc It has been pointed out by Hemacandra[1] that the period of time, called *śīrṣaprahelikā*, will be represented in terms of the period, called *pūrvi* (=8,400,000) by a number as large as to occupy 194 "notational places" (*aṅka-sthānehi*) and this number is further stated to be equal to $(8,400,000)^{28}$

Classification of numbers

Classification of numbers into odd (*oja*) and even (*yugma*) occurs in the Jaina canonical works. This distinction is very old in India. For it occurs as early as the time of the *Vedas* (c 3000 B C.).

The Jainas do not consider unity a number [3] Such was also the case with the early Greeks [4] Further classification of numbers, amongst the Jainas proceed along their orders. Thus we have in the *Anuyogadvāra-sūtra* [5]

> "What are the numbers of calculation (*gaṇanā-saṁkhyā*)? Unity does not admit of numeration, two etc are numbers They are (classified) thus *saṁkhyeya* ("numerable"), *asaṁkhyeya* ("innumerable") and *ananta* ("infinite") What are the numerables? They are known to be of three orders, such as *jaghanyā* ("lowest"), *utkṛṣṭa* ("highest") and *ajaghanyotkṛṣṭa* ("not high not-low," that is intermediate) What are the innumerables? They are of three kinds, such as *paritāsaṁkhyeya* ("nearly innumerable"), *yuktāsaṁkhyeya* ("truly innumerable") and *asaṁkhyeyakāsaṁkhyeya* ("innumerably innumerable") What are the nearly innumerables? They are of three orders, such as lowest, highest and intermediate What are the truly innumerables? They are of three orders, such as lowest, highest and intermediate What are the enumerably innumerables? They are of three orders, such as lowest, highest and intermediate What are the infinites? They are of three kinds, such as *paritānanta* ("nearly infinite"), *yuktānanta*) ("truly infinite") and *anantānanta* ("infinitely infinite") What are the nearly infinites? They are of three orders, such as lowest, highest and intermediate. What are the truly infinites? They are of three orders, such as lowest, highest and intermediate What are the infinitely infinites? They are of two orders, such as lowest and intermediate Which is the lowest numerable? It is the integer (*rupa*) two After that are the intermediate numerables until highest numeral is reached"

[1] *Anuyogadvāra-sūtra*, sūtra 116

[2] *Ibid*, Sūtra 114 (com) Compare *Samavāyāṅga sūtra*, Sūtra 84, *Jambudvīpaprajñapti*, Sūtra 18

[3] एको गननासंख्यां न उपेति।

[4] Smith, *History of Mathematics*, II, pp 26 ff.

[5] *Anuyogadvāra-sūtra*, Sūtra 146

Now the highest numerable has been defined in the work thus Consider a certain trough which is of the size of the Jambudvipa whose diameter is 100,000 *yojana* and whose circumference is 316,227 *yojana* 3 *gavyuti* 128 *dhanu* 13½ *angula* and a little over. Fill it up with white mustard seeds counting them one after another. Continue in this way to fill up with mustard seeds other troughs of the sizes of the various lands and seas of the Jaina cosmography. Still it is difficult to reach the highest number amongst the numerables So the highest numerable number of the early Jainas corresponds to what is called Alef-zero in modern mathematics For numbers beyond that *Anuyogadvāra-sūtra* further proceeds

By adding unity to the highest 'numerable,' the lowest 'nearly innumerable' is obtained After that are the intermediate numbers until the highest 'nearly innumerable' is reached. Which is the highest 'nearly innumerable'? The lowest 'nearly innumerable' number multiplied by the lowest 'nearly innumerable' number and then diminished by unity will give the highest 'nearly innumerable' number Or the lowest 'truly innumerable' number diminished by unity gives the highest 'nearly innumerable' number Which is the lowest 'truly innumerable? The lowest 'truly innumerable' is obtained by multiplying the lowest 'nearly innumerable' number by itself, or by adding unity to the highest 'nearly innumerable' number This number is also equivalent to *Âvali* After that are the intermediate numbers until the highest 'truly innumerable' number is reached Which is the highest 'truly innumerable' number? It is the lowest 'truly innumerable' number multiplied by the *Âvali* and then diminished by unity; or the lowest 'innumerably innumerable' number decreased by unity Which is the lowest innumerably innumerable number? It is the lowest 'truly innumerable' multiplied by *Âvale* or the highest 'truly innumerable' number increased by unity After that are the intermediate number until the highest 'innumerably innumerable' number is reached Which is the highest 'innumerably innumerable' number? It is the lowest 'innumerably innumerable' number multiplied by itself and then diminished by unity, or the lowest 'nearly infinite' number diminished by unity Which is the lowest 'nearly infinite' number? The lowest 'innumerably innumerable' number multiplied by itself or the highest 'innumerably innumerable' increased by unity After that are the intermediate numbers until the highest 'nearly infinite' is reached Which is the highest 'nearly infinite' number? The lowest 'nearly infinite' number multiplied by itself and the product decreased by unity, or the lowest 'truly infinite' decreased by unity. Which is the lowest 'truly infinite' number? The lowest 'nearly infinite number' multiplied by itself, or the highest 'nearly infinite' increased by unity It is also called the *Abhavasiddhi* After that are the intermediates until the highest 'truly infinite' is obtained. Which is the highest 'truly infinite' number? The lowest 'truly infinite' number multiplied by the *Abhavasiddhi* and diminished by unity or the lowest 'infinitely infinite' number diminished by unity Which is the lowest 'infinitely infinite' number? It is the lowest 'truly infinite' number multiplied by the *Abhavasiddhi* number,

or the highest 'truly infinite' added by unity. After that are intermediate numbers. Such are the numbers of calculation"

It will be easily recognised that the above classification can be represented by the following series

$$2\ldots N \mid (N+1) \quad \{(N+1)^2-1\} \mid (N+1)^2 \ldots \{(N+1)^4-1\} \mid$$
$$(N+1)^4 \quad \{(N+1)^8-1\} \mid (N+1)^8 \quad \{(N+1)^{16}-1\} \mid$$
$$(N+1)^{16}. \quad \{(N+1)^{32}-1\} \mid (N+1)^{32}$$

where N denotes the hightest numerable number as defined before. The series contains as recorded in the work the extreme numbers of each class and the different classes have been separated by a vertical line

It will be noticed that in the classification of numbers stated above there is an attempt to define numbers beyond Alef-zero. The theory of such numbers was fully developed by George Cantor in 1883. The fact that an attempt was made in India to define such numbers as early as the first century before the christian era, speaks highly of the speculative faculties of the ancient Jaina mathematicians

In another canonical work we find the following interesting classification of infinity (*ananta*) [1]

"Know that infinity is of five kinds, such as infinite in one direction, infinite in two directions, infinite in superficial expanse, infinite in all expanse, infinite in eternity"

Certain technical terms

We find certain interesting geometrical terms in the Jaina literature. It is said that the modern geometrical term "semi-diameter" was employed first by Boetius (*c* 510 A.D.).[2] It was unknown in the Greek Geometry. This term is found in the writings of Umāsvāti who calls it *vyāsārdha* [3] or *viṣkambhārdha*.[4] Still earlier in the *Âpastamba Śulba-sūtra* (*c* 800 B C.)[5], we have the term *ardha-vyāyāma* Every one of these terms literally means the "semi-diameter"

[1] *Sthānānga sūtra*, Sūtra 462—"अथवा पञ्चविध अनन्तं प्रज्ञप्त: तद्यथा, एकतो अनन्तं द्विधानन्तं, देशविस्तारानन्तं, सर्व्वविस्तारानन्तं, शाश्वतानन्तं।"

[2] Smith, *History*, II, pp 274-5

[3] *Jambudvīpasamāsa* of Umāsvāti, iv

[4] *Tattvārthādhigama sūtra-bhāṣya*, iv 14.

[5] vii 11

The term *jīva* for the chord of a segment of a circle and *dhanuprstha* for its arc occur in several early canonical works. But we miss in them the term *śara* for the arrow.

In the *Sūrya-prajñapti* (c 500 B C.)[1] occur the terms *sama-caturasra*, *viṣamacaturasra*, *samacatuṣkona*, *viṣamacatuskona*, *sama-cakravāla*, *viṣamacakravāla*, *cakrārdhacakravāla* and *chatrākāra*. According to Weber[2], they mean respectively "even square" (grades quadrat), "oblique square" (schiefes quadrat), "even parallelogram," "oblique parallelogram," "circle," "ellipse" "semi-circle" and "segment of a sphere"

In the *Bhagabatī-sūtra*[3] we find the geometrical figures *tryasra* ("triangle"), *caturasra* ("quadrilateral"), *āyata* ("rectangle") *vrtta* ("circle") and *parimandala* ("ellipse") Each of these is again classified into two kinds *pratara* ("plane") and *ghana* ("solid"). So that *ghana tryasra* denotes a "triangular pyramid", *ghana caturasra* a cube, *ghanāyata* a rectangular parallelopiped, *ghana vrtta* a sphere and *ghana parimandala* an elliptic cylinder. Reference to these figures occurs in other Jaina canonical works also.[4]

The circular annulus is called *valaya-vrtta* Similarly the the triangular and quadrangular annuli are respectively called *valaya-tryasra* and *valaya-caturasra*.

We find three units of measurement in terms of *angula* (" finger breadth") *sūcyangula* (" needle-like finger "), *prataranguła* (" plane finger ") and *ghanāngula* (" solid finger "). It is stated further that the " *sūcyangula* is linear and one-dimensional ; *sūcyangula* multiplied by *sūcyangula* gives *pratarāngula* and *pratarāngula* multiplied by *sūcyangula* becomes *ghanāngula*."[5] Hence those terms define respectively the units of linear, superficial and solid measure.

There is a very interesting passage in the *Anuyoagdvāra-sūtra*,[6] which describes how the representative number giving the measure

[1] Sūtra 19, 25, 100

[2] Weber, *Indische Studien*, X, p 274

[3] *Bhagabatī-sūtra*, Sūtra 724-726

[4] For instance *Jambudvipa-prajñapti*, *Jīvābhigama sūtra*, *Anuyogadvāra-sūtra*, Sūtra 144, 123

[5] *Anuyogadvāra-sūtra*, Sūtra 100, 132, 133 It occurs in other canonical works also.

[6] Sūtra 132.

of a certain quantity of things differs with the change of the unit of measurement. It is said that if the measure of a certain quantity of (liquid) substance be given by the number 256 when measured with a particular kind of unit, it will be given by 128, if measured with a unit twice as large If the unit of measurement be four-times the first one, the measure of the quantity will be 64. It is also stated that by increasing the units, the measure can be said to be 32, 16, 8, 4, 2, or 1

Some of the terms connected with the series in progression such as *ādi* for the first term, *gaccha* for the number of terms, *uttara* for the common difference and *ganita* for the sum of the series which are commonly found in later Hindu treatises on mathematics can be traced to the early Jaina canonical works Another interesting term which can be similarly traced back is *rupa* [1] It denotes "unity" also "an integer," in the later treatises on arithmetic but in treatises on algebra it has an entirely different significance as the "absolute" or "known" term in an equation It occurs also in the Bakhshâlî mathematics [2]

Arrangement of shots

In the *Bhagabatī-sūtra,*[3] occurs an enumeration of the minimum number of shots (*pradeśá*, literally meaning "spot", the commentator interprets it as meaning "globule") which can be arranged to have a certain geometrical form. Distinction has been drawn between *even* and *odd* number of shots. The result is given here in a tabular form —

Geometrical form.	Minimum number of *odd* shots	Minimum number of *even* shots
Circle	5	12
Sphere	7	32
Triangle	3	6
Triangular pyramid	35	4
Square	9	4
Cube	27	8
Line	3	2
Rectangle	15	6
Parallelopiped	45	12

[1] *Anuyogadvāra sūtra*, Sūtra 146 It occurs also in the *Jīvābhigama-sūtra*

[2] Bibhutibhusan Datta, "The Bakhshâlî Mathematics" *Bull Cal Math Soc*, Vol XXI, 1929, No 1, pp 1-60 See particularly pp. 21-23.

[3] XXV 3 (S. 726, 727)

No distinction of odd and even has been made in the number of shots that can be arranged in the form of an ellipse (*parimandala*) The minimum number is 20 and for the elliptic cylinder the minimum number is 40, a ring of 20 placed upon a ring of 20

The commentator observes in this connection इह ओजो युग्मभेदो न स्त युग्मरूपत्वेनैकरूपत्वात परिमण्डलस्येति । This shows that the original author is aware of the property of the ellipse that it is symmetrical about its either axis He has described the ellipse as "a circle of the shape of a barley coin" (*yavamadhyavṛtta*) [1]

A wrong formula.

We should conclude this imperfect sketch of the Jaina School of mathematics by drawing attention to a certain inaccurate result which has persisted among the Jainas even after more accurate results were discovered in India by scholars professing different faiths The area of a segment of a circle is taken as

$$\text{chord} \times \frac{\text{height}}{4} \times \sqrt{10}$$

This is found in the *Ganita-sāra-samgraha* [2] of Mahāvīra (850) and the *Laghu Kṣetra samāsa* [3] of Ratnesvara Sūri (1440) This formula is not correct and has probably been obtained by analogy from the rule for the area of a semi-circle, *viz.*,

$$\text{diameter} \times \frac{\text{height}}{4} \times \pi$$

Bull Cal. Math Soc., Vol. XXI, No 2, 1929

[1] *Bhagabatīsūtra*, Sūtra 725
[2] VII 70½
[3] Rule 191.

The inauguration of the Henri Poincaré institute in Paris

On November, 1928, was formally inaugurated a new Institute in Paris. It was both the official opening of a new building and the beginning of new courses of lectures, all to be a part of the Faculty of Sciences of the University of Paris.

The building is now ready but the internal arrangement and furnishing will not be ready before some time It was however considered a good thing to hold the ceremony in the building in order to attract public attention on the opening of the lectures and on the foundation of the Institute

It was desired to express the gratitude of the University of Paris towards those who had provided the necessary means The history of this Institute is brief It had been noted by the International Education Board that several opportunities had led them to give very large sums of money to different universities in Europe and that gifts to French ones had been on a much smaller scale Noting the importance of the French Mathematical School, it was thought that helping mathematics in France was perhaps one of the best ways of helping science all over the world.

The decision was taken after consultations, where Professor Trowbridge as representing at that time in Paris the International Education Board and Professor Birkhoff as a great mathematician, took decisive parts

It was decided to ask Professor Emile Borel to draw up a plan. The plan, which was approved, creates under the name of "Institut Henri Poincaré" a centre widely opened to teaching and researches concerning Mathematical Physics and Calculus of Probabilities

The new teaching positions have been given to three men.

The courses on "Physical Theories" will be delivered by Professor Léon Brillouin and M Louis de Broglie (to be distinguished from physicists of the same names, both members of the "Académie des Sciences"). Professor Léon Brillouin has made himself known by his deep researches on the theory of quanta and its applications ; and he was called last year to expound them in several universities of the United States and Canada. Dr Louis de Broglie is the creator of these Wave Mechanics which, born yesterday, play a leading part in Mathematical Physics and was the source of many works renovating their aspects.

Those who are interested in theoretical Physics will find in Paris that, if this is a very important addition, there were already (existing)

courses on this subject among which those of Professor Brillouin and Professor Langevin at the Collége de France, Professor Eugène Bloch and Professor Villat at the Sorbonne.

As to Calculus of Probability, it had already its great exponent at the Sorbonne in Professor Emile Borel His researches on this subject and his personal action have done much to revive in France the interest in this science which owes so much to French scientists such as Pascal, Fermat, Laplace, Poisson, Bienaymé, Cauchy, Coursot, Bertrand, Henri Poincaré

To Prosessor Borel's course will now be added a new course by Maurice Fréchet, formely Professor at the University of Strasbourg His theory of abstract spaces and functions has already made him known in America where he was called to expound it at the University of Chicago in 1924 summer quarter But he has, of late, devoted much attention to the Theory of Probability on which he published (in collaboration with Professor Halbwachs) " Le calcul des probabilités à la portée de tous "

Let us also recall that the applications of probabilities to social sciences are taught in the already existing " Institut de Statistique " of the University of Paris

But the action of the Henri Poincaré Institute will not be confined to the new courses It aims at being international in scope The attendance at these courses is very cosmopolitan indeed. But the Institute will also have an international staff of lecturers. In addition to the standing courses, single lectures or brief series of lectures will be given by distinguished scientists. Professors Vito Volterra of Rome and de Donder of Bruxelles have already promised their co-operation, other engagements will soon be published.

Finally, as the ever increasing numbers of lecturers and students at the Sorbonne called for new measures, it was decided to seize upon the opportunity and to erect a new building where not only the new courses but all the advanced courses on mathematics will be given and where the mathematical library will be housed The International Education Board is to contribute one hundred thousand dollars to these expenses Baron Edmond de Rothschild contributed also twenty five thousand dollars and the French Ministry for Education 300000 francs.

It is to be hoped that among those students and scholars who would like to complete in Europe their scientific education or to go on with their researches, some will remember that, thanks chiefly to American generosity, a great scientific international centre for Mathematical Physics and Calculus of Probability has been created in Paris.

1

On a Generalisation of Legendre Polynomials

BY

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1 The wellknown Rodrigue's formula for $P_n(x)$ led Appell* to consider polynomials of the type

$$\frac{d^n\left[x^n\left(1-x^2\right)^n\right]}{dx^n} \qquad \ldots\ldots\ldots \qquad (1)$$

These polynomials are peculiar in this sense that they satisfy a differential equation of the *third order*

$$x(1-x^2)y''' + 2(1-3x^2)y''$$
$$+ 3(n-1)(n+2)xy' + 2n(n+1)(n+2)y = 0 \qquad \ldots \qquad (2)$$

The object of the present paper is to study a more general class of functions defined by the expression

$$\lambda_{n,\mu,\nu} = \frac{d^n}{dx^n}\, x^\mu \left(\frac{1}{x} - x\right)^\nu \qquad \ldots \qquad (3)$$

where μ, ν are arbitrary constants

The above evidently includes Legendre polynomials, for we have

$$\lambda_{n,n,n} = (-1)^n 2^n n!\, P_n \qquad \ldots \qquad (4)$$

The polynomials (1) follow from (3) on putting $\mu = 2n$, $\nu = n$

* *Archiv der Math und Phys*, 3rd Series, 1901, pp 69-71.

2 The following recurrence formulae hold for the functions λ defined in (3).

$$\lambda_{n,\mu,\nu,}=(\mu-\nu)\,\lambda_{n-1,\mu-2,\nu-1}-(\mu+\nu)\lambda_{n-1,\mu\,\nu-1}, \quad \ldots \quad (5)$$

$$\lambda_{n,\mu+1,\nu}=x\lambda_{n,\mu,\nu}+n\lambda_{n-1,\mu,\nu} \quad . \quad (6)$$

$$\lambda_{n,\mu+1,\nu+1}=(1-x^2)\lambda_{n,\mu,\nu}-2nx\lambda_{n-1,\mu,\nu}-n(n-1)\lambda_{n-2,\mu,\nu} \quad (7)$$

I proceed now to obtain the differential equation satisfied by $\lambda_{n,\mu,\nu}$ and for this purpose I shall use the above recurrence formulae

From (5)

$$\lambda_{n,\mu+1,\nu+1}=(\mu-\nu)\lambda_{n-1,\mu-1,\nu}-(\mu+\nu+2)\lambda_{n-1,\mu+1,\nu}$$

$$=(\mu-\nu)\lambda_{n-1,\mu-1,\nu}$$

$$-(\mu+\nu+2)\left\{x\lambda_{n-1,\mu,\nu}+(n-1)\lambda_{n-2,\mu,\nu}\right\} \text{ by (6)}$$

(7) therefore gives

$$(\mu-\nu)\lambda_{n-1,\mu-1,}=(1-x^2)\lambda_{n,\mu,\nu}+(\mu+\nu+2-2n)x\lambda_{n-1,\mu,\nu}$$

$$+(n-1)(\mu+\nu+2-n)\,\lambda_{n-2,\mu,\nu} \quad \ldots \quad (8)$$

Applying (6) in (8) we get

$$(\mu-\nu)\,\lambda_{n-1,\mu-1,\nu}=(1-x^2)\left\{x\lambda_{n,\mu-1,\nu}+n\lambda_{n-1,\mu-1,\nu}\right\}$$

$$+(\mu+\nu+2-2n)x\left\{x\lambda_{n-1,\mu-1,\nu}+(n-1)x\lambda_{n-2,\mu-1,\nu}\right\}$$

$$+(n-1)(\mu+\nu+2-n)\left\{x\lambda_{n-2,\mu-1,\nu}+(n-2)\,\lambda_{n-3,\mu-1,\nu}\right\}.$$

On changing μ into $\mu+1$ and n into $n+3$, the above becomes

$$x(1-x^2)\lambda_{n+3,\mu,\nu}+\left\{(n-\mu+\nu+2)+(\mu+\nu-3n-6)x^2\right\}\lambda_{n+2,\mu,\nu}$$

$$+(n+2)(2\mu+2\nu-3n-3)x\lambda_{n+1,\mu,\nu}+(n+1)(n+2)(\mu+\nu-n)\lambda_{n,\mu,\nu}=0$$

Hence $\lambda_{n,\mu,\nu}$ satisfies the differential equation

$$x(1-x^2)y'''+\left\{a+2-(b+6)x^2\right\}y''$$

$$+(n+2)(3n-2b-3)xy'+(n+1)(n+2)(2n-b)y=0, \quad .. \quad (9)$$

where $\quad a=n-\mu+\nu,$

$\quad\quad b=3n-\mu-\nu$

Putting $\mu=2n$, $\nu=n$, (2) follows from (9)

If μ and ν be equal then (8) shows that $\lambda_{n,\mu,\mu}$ satisfies the differential equation of the second order

$$(1-x^2)y''+2x(\mu-n-1)y'+(n+1)(2\mu-n)y=0 \quad ... \quad (10)$$

whence Legendre equation follows on putting $\mu=n$

3. It is easy to see that $\lambda_{n,\mu,\nu}$ is of the form

$$x^{\mu-2n}\left(\frac{1}{x}-x\right)^{\nu-n}\Big\{p_{n,0}+p_{n,1}x^2+p_{n,2}x^4+$$

$$\ldots\ldots+p_{n,n}x^{2n}\Big\} \quad ... \quad (11)$$

where the p's are rational integral functions of μ, ν

$\lambda_{n,\mu,\nu}$ gives therefore a polynomial of degree n in x^2 on being multiplied by

$$x^{2n-\mu}\left(\frac{1}{x}-x\right)^{n-\nu}$$

This reminds us of the biorthogonal polynomial U_n * of nth degree in x^p defined by

$$x^{-\lambda}\left(1-\frac{x^p}{a^p}\right)^{-\mu}\frac{d^n}{dx^n}\left[x^{n+\lambda}\left(1-\frac{x^p}{a^p}\right)^{n+\mu}\right]$$

where $a>0$, $\lambda>-1$, $\mu>-1$ and $p=$ a positive integer

Let us denote our polynomial

$$x^{2n-\mu}\left(\frac{1}{x}-x\right)^{n-\nu}\lambda_{n,\mu,\nu} \text{ a particular case of } U_n \text{ by } L_{n,\mu,\nu}$$

then from (5) we have

$$L_{n,\mu,\nu}=(\mu-\nu)\,L_{n-1,\mu-2,\nu-1}-(\mu+\nu)x^2L_{n-1,\mu,\nu-1} \quad \cdots \quad (12)$$

By means of this formula we can calculate these polynomials successively The following formula, however, supplies a more convenient method

Let us write

$$L_{n,\mu,\nu}=\lambda_{0,2n-\mu,n-\nu}\,\lambda_{n,\mu,\nu}$$

$$\text{Then } \frac{d}{dx}L_{n,\mu,\nu}=\lambda_{0,2n-\mu,n-\nu}\,\lambda_{n+1,\mu,\nu}$$

$$+\lambda_{1,2n-\mu,n-\nu}\,\lambda_{n,\mu,\nu}$$

whence

$$x\,(1-x^2)L'_{n,\mu,\nu}=L_{n+1,\mu,\nu}+L_{n,\mu,\nu}\,L_{1,2n-\mu,n-\nu} \quad \cdots \quad (13)$$

Now, expressing $L_{n,\mu,\nu}$ and $L_{n+1,\mu,\nu}$ respectively in the forms

$$p_{n,0}+p_{n,1}\,x^2+p_{n,2}\,x^4+\ \cdot\ +p_{n,r}x^{2r}+\ \cdots +p_{n,n}\,x^{2n}$$

and $p_{n+1,0}+p_{n+1,1}\,x^2+p_{n+1,2}\,x^4+\quad +p_{n+1,r}\,x^{2r}+$

$$+\,p_{n+1,n+1}\,x^{2n+2}$$

* See Angelesco—"On certain biorthogonal polynomials," *C R* 176, (1923), pp 1531-1533

and by applying (13) we get

$$p_{n+1,r}=(2r-a)\,p_{n,r}-(2r-2-b)\,p_{n,r-1} \tag{14}$$

where, as before,

$$a=n-\mu+\nu,$$

$$b=3n-\mu-\nu$$

4 To obtain the differential equation satisfied by

$L_{n,\mu,\nu}$ we put

$$y=z\lambda_{0,\mu-2n,\ \nu-n}$$

in (9),

The transformed equation then becomes

$$x^3(1-x^2)^3\ z'''+x^2(1-x^2)^2(A+Bx^2)z''$$

$$+x(1-x^2)(C+Dx^2+Ex^4)z'+x^2(F+Gx^2+Hx^4)z=0 \quad . \tag{15}$$

where $A=a+2+3q_{1,0}$

$B=3q_{1,1}-b-6,$

$C=3q_{2,0}+2(a+2)q_{1,0},$

$D=3q_{2,1}+2(a+2)q_{1,1}-2(b+6)q_{1,0}+(n+2)(3n-3-2b),$

$E=3q_{2,2}-2(b+6)q_{1,1}-(n+2)(3n-3-2b),$

$F=q_{3,1}+(a+2)q_{2,1}-(b+6)q_{2,0}+(n+2)(3n-3-2b)q_{1,0}$

$\quad+(n+1)(n+2)(2n-b),$

$G=q_{3,2}-(b+6)q_{2,1}+(a+2)q_{2,2}$

$\quad+(n+2)(3n-3-2b)(q_{1,1}-q_{1,0})-2(n+1)(n+2)(2n-b),$

$H=q_{3,3}-(b+6)q_{2,2}-(n+2\ (3n-3-2b)q_{1,1}$

$\quad+(n+1)(n+2)(2n-b).$

In the above $q_{m,r}$ is the co-efficient of x^{2r} in the polynomial

$$L_{m,\mu-2n,\,\nu-n}$$

By (14) we have

$$q_{m+1,r}=(2r-m-a)q_{m,r}-(2r-2-3m-b)q_{m,r-1} \qquad . \qquad (16)$$

whence $q_{1,0}=-a$, $q_{1,1}=b$,

$$q_{2,0}=a(a+1),\ q_{2,1}=(1-a)q_{1,1}+(3+b)q_{1,0}\quad q_{2,2}=b(b+1),$$

$$q_{3,1}=-aq_{2,1}+(b+6)q_{2,0},\ q_{3,2}=(2-a)q_{2,2}+(b+1)q_{2,1},$$

$$q_{3,3}=(b+2)q_{2,2}$$

5 In (15) let us put

$$z=p_{n,0}+p_{n,1}\,x^2+\,..+p_{n,r}\,x^{2r}+\ldots+p_{n,n}\,x^{2n},$$

then equating the co-efficient of x^{2r+2} to zero we obtain the relation

$$\alpha_r p_{n,r+1}+\beta_r\,p_{n,r}+\gamma_r p_{n,r-1}+\delta_r p_{n,r-2}=0\,, \qquad .. \qquad (17)$$

$$\text{where } \alpha_r=2(r+1)\,C+2(r+1)(2r+1)A+4r(r+1)(2r+1),$$

$$\beta_r=F+2r(D-C)+2r(2r-1)(B-2A)-6r(2r-1)(2r-2),$$

$$\gamma_r=G+2(r-1)(E-D)+2(r-1)(2r-3)(A-2B)$$

$$+6(r-1)(2r-3)(2r-4),$$

$$\delta_r=H-2(r-2)\,E+2(r-2)(2r-5)B-2(r-2)(2r-5)(2r-6).$$

Putting $r=n+2$ in (17) we get

$$\delta_{n+2}\,p_{n,n}=0.$$

Thus H, E, B satisfy the relation

$$H-2n\,E+2n(2n-1)B-2n(2n-1)(2n-2)=0 \qquad .. \qquad (18)$$

By means of (17) we can successively calculate the p's in $L_{n,\mu,\nu}$ starting from the value of $p_{n,0}$, which is known from (14) to be

$$(-1)^n\,(\nu-\mu)(\nu-\mu+1)(\nu-\mu+2)\ldots\ldots(\nu-\mu+n-1)$$

The calculation may also be started from the end, for we have from (14)

$$p_{n,n}=(-1)^n\ (\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)\ ..\ \ (\mu+\nu-n+1)$$

In the case of Appell's polynomials (1) the relation (17) is much simpler

6 We give below a *recurrence formula for Appell's polynomials* Let us denote the nth polynomial

$$\lambda_{n,2n,n} \text{ or } L_{n,2n,n} \text{ by } n!\,A_n,$$

then from (5) we have

$$n!\,(A''_{n+1}-A''_n)=-3\lambda_{n+2,\,2(n+1),\,n} \qquad . \quad (19)$$

Now from (6)

$$\lambda_{n+2,\,2(n+1),n}=x\lambda_{n+2\ \ 2n+1,\ \ n}+(n+2)\lambda_{n+1,\ 2n+1,\ n}$$

$$=n!\{x^2A''_n+2(n+2)xA'_n+(n+1)(n+2)A_n\}$$

(19) therefore gives

$$A''_{n+1}=(1-3x^2)A''_n-6(n+2)x\ A'_n-3(n+1)(n+2)A_n, \qquad . \quad (20)$$

which in combination with the differential equation (2) yields many other recurrence formulae [1]

Similarly for Legendre polynomials we may deduce the relation

$$P'_n=x\,P'_{n-1}+n\,P_{n-1} \qquad \ldots \quad (21)$$

by means of the general formulae (5)-(7)

[1] *E,g* , (*i*) $6x^2\,(1-x^2)\,A''_n=3nx\,A'_{n+1}+2(n+1)(n+2)\ \ A_{n+1}$

$$-6\ (n+2)\ x(1-2x^2)\ A'_n-2(n+1)(n+2)(1-3x^2)A_n,$$

(*ii*) $3(n+1)\ x\ A'_{n+2}+2(n+2)(n+3)\ A_{n+2}-3x\ \{(8n+6)-(7n+12)x^2\}\ A'_{n+1}$

$-4(n+2)^2(1-3x^2)\ A_{n+1}+6(n+2)x(1+x^2)\ A'_n+2(n+1)(n+2)(1+3x^2)\ A_n=0,$

7. The differential equation (9) possesses three independent particular integrals, one of which is $\lambda_{n,\mu,\nu}$ when n is restricted to a positive integer The nature of the other two solutions has been studied by Humbert* in connection with the differential equation (2) which is only a particular case of (9).

My best thanks are due to Prof Ganesh Prasad for constant encouragement

* "Sur les equation de Didon," *Nouvelles Annales*, 4th series, Vol XIX, pp 43-451

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ON SOME GENERALISATIONS OF JENSEN'S INEQUALITY.

BY

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(*Read, December 29, 1928.*)

1. The object of the present paper is to generalise Jensen's inequality so as to include in it any number of sets of positive numbers a_ν, b_ν, c_ν, etc. Jensen has treated with only one set of positive numbers. His inequality in its classical form is

$$(1) \qquad \left\{ \sum_1^n a_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \sum_1^n a_\nu^q \right\}^{\frac{1}{q}} \qquad \text{for } p>q$$

or

$$(2) \qquad \left\{ \frac{1}{n} \sum_1^n a_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \frac{1}{n} \sum_1^n a_\nu^q \right\}^{\frac{1}{q}} \qquad \text{for } p<q$$

where $a_\nu \geq 0$, and p and q are positive numbers.

My first generalisation is that expressed by Theorem I below and is deduced from Jensen's inequality (1) by the application of the Cauchy-Holder inequality.* In that theorem I have treated with two sets of positive numbers a_ν, b_ν under a further condition, *viz.*, $p^{-1}+q^{-1}=1$.

From Jensen's inequality (1) we can deduce

$$(3) \qquad \left\{ \sum_1^n a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \sum_1^n a_\nu^q \right\}^{\frac{1}{q}} \left\{ \sum_1^n b_\nu^q \right\}^{\frac{1}{q}} \text{ for } p>q,$$

but I deduce in Theorem I that

$$(4) \qquad \left\{ \sum_1^n a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \sum_1^n a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \sum_1^n b_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

where $p>q$ and $p^{-1}+q^{-1}=1$, p and q being positive numbers.

* O. Hölder, Ueber einen Mittelwertsatz (*Nachrichten Ges. Wiss. Göttingen*, 1889, pp. 38-47).

(4) follows immediately from (3) only if $p>q^2$, but I do not impose any such limitation on p and q^2 so that p may or may not be greater than q^2

In § 1, I give five theorems dealing with the generalisations of (1), the first theorem being the above-mentioned one and the other theorems being further generalisations of Theorem I § 2 is devoted to the consideration of an important generalisation of Theorem I Cooper* has given a generalisation of (1) in the form

$$(5) \qquad \Psi^{-1}\{\Sigma\, \Psi(a_\nu)\} \leq \Phi^{-1}\{\Sigma\, \Phi(a_\nu)\}$$

where $\Psi(x)$ and $\Phi(x)$ are monotone (in the same sense), continuous, unbounded functions of x in $x \geq 0$ and $\Psi(x)/\Phi(x)$ increases continuously. My Theorem VI deals with two sets of numbers a_ν and b_ν, and includes (5) as a particular case. § 3 contains generalisations of Jensen's inequality (2) analogous to those treated in § 1 § 4 and § 5 are devoted to the applications of previous results to positive integrable functions within definite ranges of integration

In what follows, all the quantities a_ν, b_ν, c_ν, etc, p and q, and all the functions considered, are taken to be positive

Where no confusion is apprehended, we write Σ to denote $\sum\limits_1^n$

It is believed that the results of this paper are new

I take this opportunity to express my best thanks to Dr Ganesh Prasad for the kind interest he took in the course of the preparation of this paper

§ 1.

2 Theorem I *If a_ν and b_ν denote two sets of numbers such that $a_\nu \geq 0$, $b_\nu \geq 0$ for $\nu=1, 2, 3 \ldots n$, then will*

$$(1\ 1) \qquad \{\Sigma\, a_\nu^p\, b_\nu^p\}^{\frac{1}{p}} \leq \{\Sigma\, a_\nu^{q^2}\}^{\frac{1}{q^2}} \{\Sigma\, b_\nu^{q^2}\}^{\frac{1}{q^2}}$$

where $p>q$ and $p^{-1}+q^{-1}=1$

* R Cooper—"Notes on certain inequalities," *Journal of the London Mathematical Society*, 2 (1927) pp 159-163.

Proof

We have by Jensen's inequality (1)

$$\left\{ \Sigma\, a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \Sigma\, a_\nu^q\, b_\nu^q \right\}^{\frac{1}{q}}$$

$$\leq \left[\left\{ \Sigma\, (a_\nu^q\,)^p \right\}^{\frac{1}{p}} \left\{ \Sigma\, (b_\nu^q\,)^q \right\}^{\frac{1}{q}} \right]^{\frac{1}{q}}$$

(by Cauchy-Holder inequality)

$$\leq \left\{ \Sigma\, a_\nu^{pq} \right\}^{\frac{1}{pq}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

(I 1)
$$\leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}},$$

by Jensen's inequality (1)

Remark—As has been noticed in Art 1, in this result p need not be greater than q^{11}, it is sufficient if p be greater than q with the condition newly imposed, *viz* , $p^{-1}+q^{-1}=1$

Illustrations—

Take $p=\frac{5}{2}$, then $q=\frac{5}{3}$ and $q^2=\frac{25}{9}$ so that p is not greater than q^2 and as such Jensen's inequality is not applicable, but my generalisation * is.

Similarly taking $p=\frac{7}{3}$, $\frac{9}{4}$, $\frac{11}{5}$, $\frac{15}{7}$, etc , it is easily seen that in all such cases p is not greater than q^2 The cases in which p is greater than or equal to q^2 are, as already stated, easily deducible from (1).

3 The result of the above theorem is easily extended to any number of sets in the following theorem.

THEOREM II. *If* a_ν, b_ν, c_ν, . . k_ν *denote m sets of positive numbers, then will*

$$\left\{ \Sigma\, a_\nu^p\, b_\nu^p\, c_\nu^p \quad . \; j_\nu^p\, k_\nu^p \right\}^{\frac{1}{p}}$$

* Theorem I also holds for the generalised condition $p^{-1}+q^{-1} \geq 1$

$$(1\,2)\qquad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}}$$

$$\left\{ \Sigma\, j_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}}$$

$$(1\,3)\qquad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}} \quad \left\{ \Sigma\, k_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

where $p > q$ *and* $p^{-1} + q^{-1} = 1$

Proof.

We have

$$\left\{ \Sigma\, a_\nu^p\; b_\nu^p \;\ldots\; k_\nu^p \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \Sigma\, a_\nu^q\; b_\nu^q \;\ldots k_\nu^q \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \Sigma\, a_\nu^{pq} \right\}^{\frac{1}{pq}} \left\{ \Sigma\, b_\nu^{q^2}\, c_\nu^{q^2} \ldots\; k_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

(by Cauchy-Holder inequality)

$$(1{,}4)\qquad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2}\, c_\nu^{q^2} \;\ldots\; .k_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

[by Jensen s inequality (1)]

$$(1\,3)\qquad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}} \ldots\ldots \left\{ \Sigma\, k_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

Again by repeated applications of Cauchy Holder and Jensen's inequalities in (1 4) we get,

$$\left\{ \Sigma\, a_\nu^p\, b_\nu^p\, c_\nu^p \quad \cdot\, j_\nu^p\, k_\nu^p \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}} \left\{ \Sigma\, c_\nu^{q^3} \cdot \ldots k_\nu^{q^3} \right\}^{\frac{1}{q^3}}$$

$$\leq \quad \ldots \qquad \ldots$$

(1 2) $$\leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}} \ldots$$

$$\left\{ \Sigma\, j_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}},$$

and remembering that $\left\{ \Sigma\, a_\nu^{q^n} \right\}^{\frac{1}{q^n}} \leq \left\{ \Sigma\, a_\nu^{q^{n-1}} \right\}^{\frac{1}{q^{n-1}}} \leq \ldots.$

$$\leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \leq \left\{ \Sigma\, a_\nu^{q} \right\}^{\frac{1}{q}}$$

we have (1 2) less than (1 3)

The result (1 2) admits of being put in a still more symmetric form by the above argument, which is,

$$\left\{ \Sigma\, a_\nu^p\, b_\nu^p \,. \quad .\, j_\nu^p\, k_\nu^p \right\}^{\frac{1}{p}}$$

(1 5) $$\leq \left\{ \Sigma\, a_\nu^{q} \right\}^{\frac{1}{q}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, c_\nu^{q^3} \right\}^{\frac{1}{q^3}}$$

$$\left\{ \Sigma\, j_\nu^{q^{m-1}} \right\}^{\frac{1}{q^{m-1}}} \left\{ \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}}$$

4 Next, let us consider the summations of two or more sets of composite numbers, each term in a set being composed of two or more factors The most fundamental theorem of this type can be expressed thus —

THEOREM III. *If a_ν, b_ν, c_ν and d_ν denote four sets of positive numbers for $\nu=1, 2, 3, \ldots n$, then will*

$$\left\{ \Sigma\left(a_\nu^p\, b_\nu^p + c_\nu^p\, d_\nu^p \right)\right\}^{\frac{1}{p}}$$

$$(1\,6) \quad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}} + \left\{ \Sigma\, c_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, d_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

for $p>q$ and $p^{-1}+q^{-1}=1$

Proof

$$\left\{ \Sigma\left(a_\nu^p\, b_\nu^p + c_\nu^p\, d_\nu^p \right)\right\}^{\frac{1}{p}}$$

$$\leq \left\{ \Sigma\left(a_\nu\, b_\nu + c_\nu\, d_\nu \right)^p \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \Sigma\, a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} + \left\{ \Sigma\, c_\nu^p\, d_\nu^p \right\}^{\frac{1}{p}}$$

(by Minkowski's inequality)

$$(1\,6) \qquad \leq \left\{ \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}} + \left\{ \Sigma\, c_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \Sigma\, d_\nu^{q^2} \right\}^{\frac{1}{q^2}},$$

by Theorem I

5 In the above theorem, I have dealt with only four sets of positive numbers in two composite sets This result can, however, be extended to any number of sets Each term in the above, is composed of two numbers and the summation of two such composite sets has been considered Each term however, may be composed of any number of

factors, and the summation be considered of m such composite sets In the following theorems, I will first prove it for m sets with two factors, and then generalise the result for m sets with k factors

For the first case, let us consider the sum

$$\sum_{\nu=1}^{n} (a_\nu b_\nu + c_\nu d_\nu \quad \text{to } m \text{ such terms})$$

which can be put conveniently in a more symmetrical form

$$\sum_{\nu=1}^{n} \sum_{\mu=1}^{m} a_{\mu\nu} b_{\mu\nu}$$

Then we will have to consider the sum

$$\left\{ \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{m} a_{\mu\nu}^{p} b_{\mu\nu}^{p} \right) \right\}^{\frac{1}{p}}$$

and this is easily seen to be

$$\leq \left\{ \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{m} a_{\mu\nu} b_{\mu\nu} \right)^{p} \right\}^{\frac{1}{p}}$$

which again $\leq \sum_{\mu=1}^{m} \left\{ \sum_{\nu=1}^{n} a_{\mu\nu}^{p} b_{\mu\nu}^{p} \right\}^{\frac{1}{p}}$

(by Minkowski's inequality)

(1 7) $$\leq \sum_{\mu=1}^{m} \left\{ \left(\sum_{\nu=1}^{n} a_{\mu\nu}^{q_1} \right)^{\frac{1}{q_1}} \left(\sum_{\nu=1}^{n} b_{\mu\nu}^{q_2} \right)^{\frac{1}{q_2}} \right\},$$

by Theorem I

Thus we get

THEOREM IV *If* $a_{\mu\nu} \geq 0$, $b_{\mu\nu} \geq 0$ *for* $\mu = 1, 2, 3 \ldots m$ *and* $\nu = 1, 2, 3 \ldots n$, *then will*

$$\left\{ \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{m} a_{\mu\nu}^{p} b_{\mu\nu}^{p} \right) \right\}^{\frac{1}{p}}$$

$$(1\,7) \qquad \leq \sum_{\mu=1}^{m} \left\{ \left(\sum_{\nu=1}^{n} a_{\mu\nu}^{q^2} \right)^{\frac{1}{q^2}} \left(\sum_{\nu=1}^{n} b_{\mu\nu}^{q^2} \right)^{\frac{1}{q^2}} \right\}$$

where $p>q$ *and* $p^{-1}+q^{-1}=1$

6. In general, let us consider the sum where each term is composed of k factors and thus of the form

$$a_{\mu\nu}\, b_{\mu\nu}\, c_{\mu\nu} \ldots \ldots \ldots \kappa_{\mu\nu} \qquad or\ \Pi a_{\mu\nu}$$

The corresponding theorem can be enunciated thus —

THEOREM V. *If* $a_{\mu\nu}$, $b_{\mu\nu} \ldots \quad \kappa_{\mu\nu}$ *be each* ≥ 0, *then will*

$$(1\,8) \qquad \left\{ \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{m} \Pi a_{\mu\nu}^{p} \right) \right\}^{\frac{1}{p}} \leq \sum_{\mu=1}^{m} \left\{ \Pi \left(\sum_{\nu=1}^{n} a_{\mu\nu}^{q^2} \right)^{\frac{1}{q^2}} \right\}$$

where $p>q$ *and* $p^{-1}+q^{-1}=1$.

The proof is very simple and follows immediately from Theorem II with the help of Minkowski's inequality on the lines of Theorems III and IV The result (1 8) corresponds to (1 3), those corresponding to (1 2) and (1 5) can also be similarly deduced

§ 2

7 In this section, I proceed to prove an important generalisation of the Theorem I Cooper's generalisation (5) of Jensen's inequality (1) follows from the following theorem as a particular case with one set of positive numbers It can be stated thus—

THEOREM VI *If* $\Psi(x)$ *and* $\Phi(x)$ *be two monotone, continuous and increasing functions in* $x \geq 0$ *and* $\frac{\Psi(x)}{\Phi(x)}$ *continuously increases, then will*

$$(2\,1) \qquad \Psi^{-1}\{\Sigma\, \Psi(a_\nu b_\nu)\} \leq \Phi^{-2}\{\Sigma\, \Phi^2(a_\nu)\} \quad \Phi^{-2}\{\Sigma\, \Phi^2(b_\nu)\}$$

Proof

Since $\frac{\Psi(x)}{\Phi(x)}$ increases continuously we have $\Psi(x) \succ \Phi(x)$

Put $\Psi(x) \equiv \kappa(x)\, \Phi(x)$

Then since $\Phi(r)$ is an increasing function of x, we have

$$\Phi(a_\nu \quad b_\nu \ ) \leq \Sigma \, \Phi \, (a_\nu \quad b_\nu \ )$$

$$\therefore \ a_\nu \quad b_\nu \ \leq \Phi^{-1}\{ \Sigma \, \Phi(a_\nu \ b_\nu \ )\}$$

Hence $\Sigma \, \Psi(a_\nu \ b_\nu \ ) \equiv \Sigma \, \Phi(a_\nu \ b_\nu \ ) \, \kappa(a_\nu \ b_\nu \ )$

$$\leq \Sigma \, \Phi(a_\nu \ b_\nu \ ) \quad \kappa[\Phi^{-1}\{\Sigma \, \Phi(a_\nu \ b_\nu \ )\}]$$

$$\leq \Sigma \, \Phi \, (a_\nu \ b_\nu \ ) \frac{\Psi[\Phi^{-1}\{\Sigma \, \Phi(a_\nu \ b_\nu \ )\}]}{\Phi[\Phi^{-1}\{\Sigma \, \Phi(a_\nu \ b_\nu \ )\}]}$$

$$\leq \Psi \, [\Phi^{-1}\{\Sigma \, \Phi(a_\nu \ b_\nu \ )\}$$

(2 2) $\therefore \ \Psi^{-1}\{\Sigma \, \Psi(a_\nu \ b_\nu \ )\} \leq \Phi^{-1}\{\Sigma \, \Phi(a_\nu \ b_\nu \ )\}$

$$\leq \Phi^{-1}\{\Sigma \, \Phi(a_\nu \ ) \quad \Phi(b_\nu \ )\}$$

$$\leq \Phi^{-1}[\Phi^{-1}\{\Sigma \, \Phi^2(a_\nu \ )\} \quad \Psi^{-1}\{\Sigma \, \Psi\Phi(b_\nu \ )\}]$$

(by Cooper's generalisation of Hölder's inequality*)

(2 1) $$\leq \Phi^{-2}\{\Sigma \, \Phi^2(a_\nu \ )\} \quad \Phi^{-2}\{\Sigma \, \Phi^2(b_\nu \ )\}$$

Cooper's result† (5) is easily deduced from (2 2) In the result (2 1) the functions Φ and Ψ are modified by the restrictions laid down by Cooper and Hardy, namely that either

$$\Phi(x) = x\phi(x), \qquad\qquad \Psi(x) = x\psi(x),$$

or

$$\Phi(x) = \int_0^x \phi(t)dt, \qquad\qquad \Psi(x) = \int_0^x \psi(t)dt,$$

$\phi(x)$ and $\psi(x)$ being continuous increasing functions of x, differentiable everywhere, which vanish with x and are inverse to one another, so that ϕ and ψ are of the form Ax^a

* R Cooper—Note on Cauchy-Hölder inequality, *Proc London Math Soc* 26 (1927), 415 432

—Note on Cauchy-Hölder inequality, *Jour London Math Soc*, 3 (1928), 8 9

G H Hardy—Remarks on three recent notes in the Journal, *Jour London Math. Soc*, 3 (1928), 166-169

Francis and Littlewood—*Examples in Infinite Series.*

† See p 2

§ 3

8. In the foregoing pages I have considered generalisations of Jensen's inequality in the form (1). Now I propose to give similar generalisations from the form (2). The results are similarly deduced. The theorem corresponding to the Theorem I can be stated thus:

THEOREM VII. *If $a_\nu \geq 0$, $b_\nu \geq 0$ denote two sets of positive numbers, then will*

$$(3\cdot 1) \qquad \left\{ \frac{1}{n} \Sigma\, a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

where $p<q$ *and* $p^{-1}+q^{-1}=1$

Proof.

We have $\left\{ \frac{1}{n} \Sigma\, a_\nu^p\, b_\nu^p \right\}^{\frac{1}{p}} \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^q\, b_\nu^q \right\}^{\frac{1}{q}}$

$$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{pq} \right\}^{\frac{1}{pq}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

(by Cauchy-Hölder inequality)

$$(3\cdot 1) \qquad \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

$$(3\cdot 2) \qquad \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^n} \right\}^{\frac{1}{q^n}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^n} \right\}^{\frac{1}{q^n}}.$$

It is to be observed in this connection that unlike the previous cases in this theorem as also in what follows p is not greater but less than q

9 Next let us generalise the above to include any number of sets The results are slightly different from those of the previous case, and from what has been shewn before these appear evident,

THEOREM VIII *If* $a_\nu \geq 0$, $b_\nu \geq 0$, $c_\nu \geq 0$. $\ldots k_\nu \geq 0$ *denote m sets of positive numbers, then will*

$$\left\{ \frac{1}{n} \Sigma\, a_\nu^p\, b_\nu^p\, c_\nu^p \ldots\ldots\ldots j_\nu^p\, k_\nu^p \right\}^{\frac{1}{p}}$$

(3 3) $$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}}$$

$$\left\{ \frac{1}{n} \Sigma\, j_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \frac{1}{n} \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}}$$

(3 4) $$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^m} \right\}^{\frac{1}{q^m}} \cdot \left\{ \frac{1}{n} \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}}$$

where $p<q$, *and* $p^{-1}+q^{-1}=1$

Proof

$$\left\{ \frac{1}{n} \Sigma\, a_\nu^p\, b_\nu^p\, c_\nu^p \ldots\ldots\ldots j_\nu^p\, k_\nu^p \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^2} c_\nu^{q^2} \,.\, k_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

[by (3 1)]

$$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}} \left\{ \frac{1}{n} \Sigma\, c_\nu^{q^3} \ldots k_\nu^{q^3} \right\}^{\frac{1}{q^3}},$$

by Cauchy-Holder and Jensen's inequalities

Proceeding with repeated operations of the above inequalities we have ultimately the above

(3 3) $$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \ldots \left\{ \frac{1}{n} \Sigma\, j_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \frac{1}{n} \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}}$$

(3 4) $$\leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^m} \right\}^{\frac{1}{q^m}} \ldots \left\{ \frac{1}{n} \Sigma\, k_\nu^{q^m} \right\}^{\frac{1}{q^m}},$$

by (3·2)

It is also easily seen that (3 3) is

$$(3\cdot5) \quad \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^n} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^3} \right\}^{\frac{1}{q^3}} \cdots \left\{ \frac{1}{n} \Sigma\, j_\nu^{q^m} \right\}^{\frac{1}{q^m}} \left\{ \frac{1}{n} \Sigma\, k_\nu^{q^{m+1}} \right\}^{\frac{1}{q^{m+1}}}$$

$$(3\ 6) \quad \leq \Pi \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^{m+1}} \right\}^{\frac{1}{q^{m+1}}}$$

10 As in cases of Theorems III-V, the results of the Theorems VII and VIII admit of being applied to composite sets of terms The lines of proof are based on similar lines and it will suffice to formulate the theorems only. I shall prove for the case where each term is com posed of m factors and summation of r such composite sets are dealt with

THEOREM IX *If* a_ν , b_ν , c_ν *and* d_ν *denote four sets of positive numbers, then will*

$$\left\{ \frac{1}{n} \Sigma \left(a_\nu^p\, b_\nu^p + c_\nu^p\, d_\nu^p \right) \right\}^{\frac{1}{p}}$$

$$(3\cdot7) \quad \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q} \right\}^{\frac{1}{q^2}} + \left\{ \frac{1}{n} \Sigma\, c_\nu^{q^2} \right\}^{\frac{1}{q^2}} \left\{ \frac{1}{n} \Sigma\, d_\nu^{q^2} \right\}^{\frac{1}{q^2}}$$

$$(3\ 8) \quad \leq \left\{ \frac{1}{n} \Sigma\, a_\nu^{q^n} \right\}^{\frac{1}{q^n}} \left\{ \frac{1}{n} \Sigma\, b_\nu^{q^n} \right\}^{\frac{1}{q^n}} + \left\{ \frac{1}{n} \Sigma\, c_\nu^{q^n} \right\}^{\frac{1}{q^n}} \left\{ \frac{1}{n} \Sigma\, d_\nu^{q^n} \right\}^{\frac{1}{q^n}}$$

where $p<q$ *and* $p^{-1}+q^{-1}=1$.

THEOREM X *If* $a_{\mu\nu}\geq 0$, $b_{\mu\nu}\geq 0$ *for* $\mu=1, 2, 3, \ldots m$, *and* $\nu=1, 2, 3,\ldots n$, *then will*

$$\left\{\frac{1}{n}\sum_{\nu=1}^{n}\left(\sum_{\mu=1}^{m}a^{p}_{\mu\nu}b^{p}_{\mu\nu}\right)\right\}^{\frac{1}{p}}$$

$$\leq \sum_{\mu=1}^{m}\left\{\frac{1}{n}\sum_{\nu=1}^{n}a^{q^2}_{\mu\nu}\right\}^{\frac{1}{q^2}}\left\{\frac{1}{n}\sum_{\nu=1}^{n}b^{q^2}_{\mu\nu}\right\}^{\frac{1}{q^2}} \tag{3 9}$$

$$\leq \sum_{\mu=1}^{m}\left\{\frac{1}{n}\sum_{\nu=1}^{n}a^{q^n}_{\mu\nu}\right\}^{\frac{1}{q^n}}\left\{\frac{1}{n}\sum_{\nu=1}^{n}b^{q^n}_{\mu\nu}\right\}^{\frac{1}{q^n}} \tag{3 10}$$

where $p<q$ *and* $p^{-1}+q^{-1}=1$

The proofs of these theorems are easily deduced

THEOREM XI *If* $a_{\nu\rho}$, $b_{\nu\rho}$, $c_{\nu\rho}$ . .. $k_{\nu\rho}$ *be each* ≥ 0 *for* $\nu=1, 2, 3,\ldots n$ *and* $\rho=1, 2, 3, \ldots r$, *then will*

$$\left\{\frac{1}{n}\sum_{\nu=1}^{n}\left(\sum_{\rho=1}^{r}\Pi a^{p}_{\nu\rho}\right)\right\}^{\frac{1}{p}}$$

$$\leq \sum_{\rho=1}^{r}\left\{\frac{1}{n}\sum_{\nu=1}^{n}a^{q^2}_{\nu\rho}\right\}^{\frac{1}{q^2}}\left\{\frac{1}{n}\sum_{\nu=1}^{n}b^{q^3}_{\nu\rho}\right\}^{\frac{1}{q^3}}\cdots\left\{\frac{1}{n}\sum_{\nu=1}^{n}k^{q^{m+1}}_{\nu\rho}\right\}^{\frac{1}{q^{m+1}}} \tag{3 11}$$

$$\leq \sum_{\rho=1}^{r}\left\{\Pi\left(\frac{1}{n}\sum_{\nu=1}^{n}a^{q^{m+1}}_{\nu\rho}\right)^{\frac{1}{q^{m+1}}}\right\} \tag{3 12}$$

where $p<q$ *and* $p^{-1}+q^{-1}=1.$

Proof. We have

$$\left\{\frac{1}{n}\sum_{\nu=1}^{n}\left(\sum_{\rho=1}^{r}\Pi a^{p}_{\nu\rho}\right)\right\}^{\frac{1}{p}}\leq\left\{\frac{1}{n}\sum_{\nu=1}^{n}\left(\sum_{\rho=1}^{r}\Pi a_{\nu\rho}\right)^{p}\right\}^{\frac{1}{p}}$$

$$\leq \sum_{\rho=1}^{r}\left\{\frac{1}{n}\sum_{\nu=1}^{n}\Pi a^{p}_{\nu\rho}\right\}^{\frac{1}{p}}$$

(by Minkowski's inequality)

$$\leq \sum_{\rho=1}^{r} \left\{ \frac{1}{n} \sum_{\nu=1}^{n} a_{\nu\rho}^{q^2} \right\}^{\frac{1}{q^2}} \cdot \left\{ \frac{1}{n} \sum_{\nu=1}^{n} k_{\nu\rho}^{q^{m+1}} \right\}^{\frac{1}{q^{m+1}}}$$

[by (3·5)]

$$\leq \sum_{\rho=1}^{r} \left\{ \Pi \left(\frac{1}{n} \sum_{\nu=1}^{n} a_{\nu\rho}^{q^{m+1}} \right)^{\frac{1}{q^{m+1}}} \right\},$$

by (3 6)

Results corresponding to the forms (3 3) and (3 4) can be easily deduced

These are the most general forms of the inequalities considered in this section and include the Theorems VII-X as particular cases

§ 4

Applications to Definite Integrals

12. The rest of the paper is devoted to the applications of the results of the foregoing theorems to positive integrable functions within definite ranges of integration.

Firstly let us consider the results of § 1, and see if they admit of being applied to integrable functions Those corresponding to § 3 will be considered in § 5

The theorem corresponding to Theorem I can be enunciated as follows, and let us see if the result holds in the case of positive integrable functions.

THEOREM XII *If $f(x)$ and $g(x)$ be two positive integrable functions in $x_1 \leq x \leq x_2$, then will*

$$\left\{ \int_{x_1}^{x_2} [f(x)g(x)]^p \, dx \right\}^{\frac{1}{p}}$$

(4 1) $$\leq \left\{ \int_{x_1}^{x_2} [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_{x_1}^{x_2} [g(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

where $p > q$ *and* $p^{-1}+q^{-1}=1$

Proof

Put $f_\nu(x)=f(x_1+\nu h)$

$(\nu=1, 2, 3, \ldots, n)$

and $g_\nu(x)=g(x_1+\nu h)$

where $h=\dfrac{x_2-x_1}{n}$

Then we have

$$\left\{\int_{x_1}^{x_2}[f(x)g(x)]^p\,dx\right\}^{\frac{1}{p}}=\operatorname*{Lt}_{h\to 0}\left\{\Sigma\, h\, f_\nu^p\, g_\nu^p\right\}^{\frac{1}{p}}$$

$$\leq \operatorname*{Lt}_{h\to 0} h^{\frac{1}{p}}\left\{\Sigma f_\nu^{q^2}\right\}^{\frac{1}{q^2}}\left\{\Sigma g_\nu^{q^2}\right\}^{\frac{1}{q^2}}, \text{ by (1 1)}$$

$$\leq \operatorname*{Lt}_{h\to 0} h^{\frac{1}{p}-\frac{2}{q^2}}\left\{\int_{x_1}^{x_2}[f(x)]^{q^2}dx\right\}^{\frac{1}{q^2}}\cdot\left\{\int_{x_1}^{x_2}[g(x)]^{q^2}dx\right\}^{\frac{1}{q^2}}$$

Now (4 1) will hold if

$\operatorname*{Lt}_{h\to 0} h^{\frac{1}{p}-\frac{2}{q^2}}$ can be shown to be equal to 1 or some positive proper fraction But as h is an infinitesimal, $\frac{1}{p}-\frac{2}{q^2}$ must be ≥ 0 This combined with the two given relations between p and q, gives $p=q=2$ Thus we get as a paticular case,

$$(4\ 2)\quad \left\{\int_{x_1}^{x_2}[f(x)g(x)]^2dx\right\}^{\frac{1}{2}}$$

$$\leq\left\{\int_{x_1}^{x_2}[f(x)]^4dx\right\}^{\frac{1}{4}}\left\{\int_{x_1}^{x_2}[g(x)]^4dx\right\}^{\frac{1}{4}}$$

where $p=q=2$,

Thus we see that the result (1 1) does not hold good generally in the case of integrable functions This is also evident from the fact that there is no simple inequality for integrals corresponding to Jensen's

inequality (1) With the help of Jensen's inequality (2) however, we can deduce an inequality in integrals analogous in form to (1) but opposite in sense and further, the range of integration is limited to (0, 1).

For we have

$$\left\{\int_{x_1}^{x_2} [f(x)]^p \, dx\right\}^{\frac{1}{p}} = \operatorname*{Lt}_{n\to\infty} \left\{\Sigma \frac{x_2-x_1}{n} f_\nu^p \right\}^{\frac{1}{p}}$$

$$\leq (x_2-x_1)^{\frac{1}{p}} \operatorname*{Lt}_{n\to\infty} \left\{\frac{1}{n} \Sigma f_\nu^q \right\}^{\frac{1}{q}}, \qquad \text{by (2),}$$

where $p<q$

$$\leq (x_2-x_1)^{\frac{1}{p}-\frac{1}{q}} \left\{\int_{x_1}^{x_2} [f(x)]^q dx\right\}^{\frac{1}{q}},$$

which can be

$$\leq \left\{\int_{x_1}^{x_2} [f(x)]^q dx\right\}^{\frac{1}{q}} \text{ only in the range } (0, 1)$$

Thus we have

$$(4\ 3) \qquad \left\{\int_0^1 [f(x)]^p dx\right\}^{\frac{1}{p}} \leq \left\{\int_0^1 [f(x)]^q dx\right\}^{\frac{1}{q}}$$

where $p<q$.

13 Now let us see if with this modification, analogous results can be obtained.

Consider the integral $\int_0^1 [f(x)g(x)]^p dx$ $(p>1)$

By the application of Cauchy-Hölder inequality, we have

$$\int_0^1 [f(x)g(x)]^p dx \leq \left\{\int_0^1 [f(x)]^{p^2} dx\right\}^{\frac{1}{p}} \left\{\int_0^1 [g(x)]^{pq} dx\right\}^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$

Hence

$$\left\{\int_0^1 [f(x)g(x)]^p\,dx\right\}^{\frac{1}{p}} \le \left\{\int_0^1 [f(x)]^{p^2}dx\right\}^{\frac{1}{p^2}} \left\{\int_0^1 [g(x)]^{pq}dx\right\}^{\frac{1}{pq}}$$

$$\le \left\{\int_0^1 [f(x)]^{q^2}dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)]^{q^2}dx\right\}^{\frac{1}{q^2}}, \quad \text{by (4 3)}$$

where $p<q$.

Thus we get

THEOREM XIII. *If $f(x)$ and $g(x)$ be two positive integrable functions in $0 \le x \le 1$, then will*

(4 4) $$\left\{\int_0^1 [f(x)\,g(x)]^p\,dx\right\}^{\frac{1}{p}}$$

$$\le \left\{\int_0^1 [f(x)]^{q^2}dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)]^{q^2}dx\right\}^{\frac{1}{q^2}}$$

where $p<q$ and $p^{-1}+q^{-1}=1$

14 As in previous cases, the above result can easily be extended to any number of integrable functions, but as regards the indices the results will correspond to the results of § 3

THEOREM XIV *If $f(x)$, $g(x)$, $\phi(x)$ $\theta(x)$ and $h(x)$ be any positive integrable functions m in number, in the interval $0 \le x \le 1$, then will*

$$\left\{\int_0^1 [f(x)\,g(x) \quad \theta(x)\,h(x)]^p\,dx\right\}^{\frac{1}{p}}$$

(4 5) $$\le \left\{\int_0^1 [f(x)]^{q^2}dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)]^{q^3}dx\right\}^{\frac{1}{q^3}} \cdots$$

$$\left\{\int_0^1 [\theta(x)]^{q^m}dx\right\}^{\frac{1}{q^m}} \cdot \left\{\int_0^1 [h(x)]^{q^m}dx\right\}^{\frac{1}{q^m}}$$

$$(4\ 6) \quad \leq \left\{ \int_0^1 [f(x)]^{q^m} dx \right\}^{\frac{1}{q^m}} \cdot \left\{ \int_0^1 [g(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

$$\cdot \quad \left\{ \int_0^1 [h(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

where, as before, $p<q$ *and* $p^{-1}+q^{-1}=1$

Proof

Put $g(x)\,\phi(x) \quad \ldots h(x)=A(x)$

$$\text{Then} \left\{ \int_0^1 [f(x)g(x) \quad h(x)]^p\, dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_0^1 [f(x) \quad A(x)]^p\, dx \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \cdot \left\{ \int_0^1 [A(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}, \text{ by } (4\ 4)$$

$$\leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x) \quad B(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

[where $A(x)=g(x)B(x)$]

$$\leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x)]^{pq^2}\, dx \right\}^{\frac{1}{pq^2}} \cdot$$

$$\left\{ \int_0^1 [B(x)]^{q^3} dx \right\}^{\frac{1}{q^3}}$$

(by Cauchy-Hölder inequality)

$$\leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x)]^{q^3} dx \right\}^{\frac{1}{q^3}} \left\{ \int_0^1 [B(x)]^{q^3} dx \right\}^{\frac{1}{q^3}}.$$

Proceeding in the same way, and by simultaneous applications of Cauchy-Holder and Jensen's inequalities we ultimately have the above

$$(4\cdot 5)\quad \leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x)]^{q^3} dx \right\}^{\frac{1}{q^3}} \cdots$$

$$\left\{ \int_0^1 [\theta(x)]^{q^m} dx \right\}^{\frac{1}{q^m}} \left\{ \int_0^1 [k(x)]^{q^m} dx \right\}^{\frac{1}{q^m}},$$

and since

$$\left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \leq \left\{ \int_0^1 [f(x)]^{q^3} dx \right\}^{\frac{1}{q^3}} \leq \cdots$$

$$\leq \left\{ \int_0^1 [f(x)]^{q^m} dx \right\}^{\frac{1}{q^m}},$$

the result (4·5) is easily seen to be

$$(4\cdot 6)\quad \leq \left\{ \int_0^1 [f(x)]^{q^m} dx \right\}^{\frac{1}{q^m}} \left\{ \int_0^1 [g(x)]^{q^m} dx \right\}^{\frac{1}{q^m}} \cdots\cdots$$

$$\cdots \left\{ \int_0^1 [k(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}.$$

Also we have the result (4·5)

$$(4\cdot 7)\quad \leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x)]^{q^3} dx \right\}^{\frac{1}{q^3}} \cdots\cdots$$

$$\left\{ \int_0^1 [k(x)]^{q^{m+1}} dx \right\}^{\frac{1}{q^{m+1}}}.$$

$$(4\cdot 8)\quad \leq \Pi \left\{ \int_0^1 [f(x)]^{q^{m+1}} dx \right\}^{\frac{1}{q^{m+1}}}.$$

15. In Theorems III–V, I have treated with composite sets of numbers. Let us now do the same with integrable functions. The theorem corresponding to Theorem III can be put thus—

THEOREM XV. *If $f(x)$, $g(x)$, $h(x)$ and $k(x)$ be four positive integrable functions in $0 \leq x \leq 1$, then will*

$$\left\{ \int_0^1 [f^p(x)\, g^p(x) + h^p(x)\, k^p(x)]\, dx \right\}^{\frac{1}{p}} *$$

$$\text{(4.9)} \qquad \leq \left\{ \int_0^1 [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [g(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

$$+ \left\{ \int_0^1 [h(x)]^{q^2} dx \right\}^{\frac{1}{q^2}} \left\{ \int_0^1 [k(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

where $p < q$ and $p^{-1} + q^{-1} = 1$.

Proof. We have

$$\left\{ \int_0^1 [f^p(x)\, g^p(x) + h^p(x)\, k^p(x)]\, dx \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \int_0^1 [f(x) g(x) + h(x)\, k(x)]^p\, dx \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \int_0^1 [f(x)\, g(x)]^p\, dx \right\}^{\frac{1}{p}} + \left\{ \int_0^1 [h(x)\, k(x)]^p\, dx \right\}^{\frac{1}{p}}$$

(by generalisation of Minkowski's inequality †)

* Here $f^p(x)$ is meant to denote $[f(x)]^p$ and not $\left(\frac{d}{dx}\right)^p f(x)$ as it often does.

† § 91. *Aufgaben und Lehrsätze aus der Analysis I Bd. XIX*
—G. Pólya und G. Szegö.

$$(4\cdot 9)\quad \leq \left\{\int_0^1 [f(x)]^{q^2} dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)]^{q^2} dx\right\}^{\frac{1}{q^2}}$$

$$+ \left\{\int_0^1 [h(x)]^{q^2} dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [k(x)]^{q^2} dx\right\}^{\frac{1}{q^2}},$$

by Theorem XIII

In this theorem, I have considered only four different functions in two groups, each consisting of two functions As in Theorems IV and V, this result also admits of generalisations in analogous forms The theorem corresponding to Theorem IV amounts to simple addition of several composite terms, each of two functions on the lefthand side and their corresponding terms on the righthand side. The proof follows easily, and is based on exactly the same lines as those of Theorem XV. It is left to the reader It, as also Theorem XV, are generalisations of Theorem XIII which they include as a particular case I shall prove it now for the more general case, namely, where each term is composed of m functions

16 In what follows $f(x)_\rho$, $\rho=1, 2, 3 \ldots r$, denotes a series of functions $f(x)_1$, $f(x)_2$, $\ldots$ $f(x)_r$, the functions themselves having no necessary relation or connection with one another

THEOREM XVI *If* $f(x)_\rho$, $g(x)_\rho$, $\ldots$ $k(x)_\rho$ *be* m *series of positive integrable functions in* $0 \leq x \leq 1$, *then will*

$$\left\{\int_0^1 \sum_{\rho=1}^{r} [f(x)_\rho \, g(x)_\rho \cdots k(x)_\rho]^p \, dx\right\}^{\frac{1}{p}}$$

$$(4\cdot 10)\quad \leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho]^{q^2} dx\right\}^{\frac{1}{q^2}} \cdots \left\{\int_0^1 [k(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

$$(4\cdot 11)\quad \leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}} \cdots\cdots \left\{\int_0^1 [k(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

where $p<q$ *and* $p^{-1}+q^{-1}=1$

Proof

We have

$$\left\{\int_0^1 \sum_{\rho=1}^{r} [f(x)_\rho\, g(x)_\rho \ldots k(x)_\rho]^p\, dx\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_0^1 \left[\sum_{\rho=1}^{r} f(x)_\rho\, g(x)_\rho \ldots k(x)_\rho\right]^p dx\right\}^{\frac{1}{p}}$$

$$\leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho\, g(x)_\rho \ldots k(x)_\rho]^p\, dx\right\}^{\frac{1}{p}}$$

(by generalisation of Minkowski's inequality)

$$(4.10) \quad \leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho]^{q^2} dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)_\rho]^{q^3} dx\right\}^{\frac{1}{q^3}} \ldots \ldots \left\{\int_0^1 [k(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

[by (4.5)]

$$(4.11) \quad \leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}} \left\{\int_0^1 [g(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}} \cdot \ldots \cdot \left\{\int_0^1 [k(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

[by (4.6)]

Also we have

$$\left\{\int_0^1 \sum_{\rho=1}^{r} [f(x)_\rho\, g(x)_\rho \ldots k(x)_\rho]^p\, dx\right\}^{\frac{1}{p}}$$

$$(4.12) \quad \leq \sum_{\rho=1}^{r} \left\{\int_0^1 [f(x)_\rho]^{q^2} dx\right\}^{\frac{1}{q^2}} \left\{\int_0^1 [g(x)_\rho]^{q^3} dx\right\}^{\frac{1}{q^3}} \cdot \ldots \ldots \left\{\int_0^1 [k(x)_\rho]^{q^{m+1}} dx\right\}^{\frac{1}{q^{m+1}}},$$

by (4.7).

This theorem is the most general one of the type considered and includes all the results of theorems XIII-XV as particular cases

§ 5

17 Unlike the previous case, the results of § 3 can all be applied to integrable functions and from what has already been discussed in the foregoing pages, the applications to positive integrable functions are quite evident The difficulties of the previous case do not in the least arise here I shall briefly discuss the case of two functions in Theorem XVII, and then generalise for any number of functions and finish this discussion with a proof for the case of series of composite functions, each term being composed of m functions

THEOREM XVII *If $f(x)$ and $g(x)$ be any two positive integrable functions in $x_1 \leq x \leq x_2$, then will*

$$\left\{\frac{1}{x_2-x_1} \int_{x_1}^{x_2} [f(x)\ g(x)]^p\, dx\right\}^{\frac{1}{p}}$$

$$(5.1) \qquad \leq \left\{\frac{1}{x_2-x_1} \int_{x_1}^{x_2} [f(x)]^{q^2} dx\right\}^{\frac{1}{q^2}}$$

$$\left\{\frac{1}{x_2-x_1} \int_{x_1}^{x_2} [g(x)]^{q^2} dx\right\}^{\frac{1}{q^2}}$$

where $p<q$ and $p^{-1}+q^{-1}=1$.

Proof

Using the notations of Theorem XII, we have

$$\left\{\frac{1}{x_2-x_1} \int_{x_1}^{x_2} [f(x)\ g(x)]^p\, dx\right\}^{\frac{1}{p}} = \underset{n\to\infty}{Lt} \left\{\frac{1}{n} \Sigma f_\nu^p\, g_\nu^p\right\}^{\frac{1}{p}}$$

$$\leq \underset{n\to\infty}{Lt} \left\{\frac{1}{n} \Sigma f_\nu^{q^2}\right\}^{\frac{1}{q^2}} \left\{\frac{1}{n} \Sigma g_\nu^{q^2}\right\}^{\frac{1}{q^2}}$$

[by (3.1)]

$$(5\cdot1) \qquad \leq \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

$$\left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [g(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

which again, as in (3 2)

$$(5\cdot2) \quad \leq \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^{q^n} dx \right\}^{\frac{1}{q^n}} \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [g(x)]^{q^n} dx \right\}^{\frac{1}{q^n}}$$

18 This result is easily extended to any number of functions and the corresponding result can be thus enunciated

THEOREM XVIII *If $f(x)$, $g(x)$, $\phi(x)$ $\quad$ $k(x)$ be any m positive integrable functions in $x_1 \leq x \leq x_2$, then will*

$$\left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)\ g(x)\ \phi(x) \quad k(x)]^p\, dx \right\}^{\frac{1}{p}}$$

$$(5\cdot3) \quad \leq \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^{q^2} dx \right\}^{\frac{1}{q^2}}$$

$$\left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [g(x)]^{q^3} dx \right\}^{\frac{1}{q^3}} \quad . \quad ..$$

$$. \quad . \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [k(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

$$(5\cdot4) \qquad \leq \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

$$\ldots \ldots \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [k(x)]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

where, as before, $p<q$ and $p^{-1}+q^{-1}=1$.

These, as also other analogous results, can be deduced very easily with the help of the previous theorems.

19. This paragraph is devoted to the consideration of series of composite functions. The cases where each series is composed of two or three functions are easy to deduce. I will prove for the most general case, namely, with r series of composite functions, each composed of m functions. This corresponds to Theorem XVI of the previous case. It is stated thus —

THEOREM XIX. *If $f(x)_\rho$, $g(x)_\rho$, . . $k(x)_\rho$ be m series of positive integrable functions in $x_1 \leq x \leq x_2$, then will*

$$\left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \sum_{\rho=1}^{r} \left[f(x)_\rho \; g(x)_\rho \; \ldots \; k(x)_\rho \right]^p dx \right\}^{\frac{1}{p}}$$

(5.5)
$$\leq \sum_{\rho=1}^{r} \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left| f(x)_\rho \right|^{q'} dx \right\}^{\frac{1}{q'}} \ldots \ldots$$

$$\ldots \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left[k(x)_\rho \right]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

(5.6)
$$\leq \sum_{\rho=1}^{r} \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left| f(x)_\rho \right|^{q^m} dx \right\}^{\frac{1}{q^m}} \ldots$$

$$\ldots\ldots \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left[k(x)_\rho \right]^{q^m} dx \right\}^{\frac{1}{q^m}}$$

where $p<q$ and $p^{-1}+q^{-1}=1$

Proof.

We have

$$\left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \sum_{\rho=1}^{r} \left[f(x)_\rho \; g(x)_\rho \ldots\ldots k(x)_\rho \right]^p dx \right\}^{\frac{1}{p}}$$

5

$$\leq \left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Big[\sum_{\rho=1}^{r} f(x)_\rho\, g(x)_\rho \quad \cdots \quad k(x)_\rho\Big]^p dx\right\}^{\frac{1}{p}}$$

$$\leq \sum_{\rho=1}^{r}\left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}[f(x)_\rho\, g(x)_\rho \quad \cdot\, k(x)_\rho]^p\, dx\right\}^{\frac{1}{p}}$$

$$(5\ 5) \qquad \leq \sum_{\rho=1}^{r}\left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}[f(x)_\rho]^{q^2} dx\right\}^{\frac{1}{q^2}}\cdots\cdots$$

$$\cdots\cdots \quad \cdots \left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2} k(x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

[by (5 3)]

$$(5\ 6) \qquad \leq \sum_{\rho=1}^{r}\left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}[\,f(x)_\rho\,]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

$$\cdots \quad \cdots \left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}[k\ x)_\rho]^{q^m} dx\right\}^{\frac{1}{q^m}}$$

by (5 4)

Results corresponding to (3 11) and (3 12) also follow easily

20 Some of the results of this paper admit of further extensions and developments which will give rise to very interesting results.

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3

FLEXURE OF A BEAM HAVING A SECTION IN THE FORM OF A RIGHT-ANGLED ISOSCELES TRIANGLE

BY

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The problems of the bending of beams by transverse terminal loads have been solved only in a few cases of sections An approximate solution of the problem for the section of the form of an isosceles triangle was given by S Timoschenko * in his paper on "The flexure analogy of Membranes" In the present paper, I have found exact values of the stresses in a uniform beam bent by a terminal load when the section is a right angled isosceles triangle.

We suppose that one end of the beam is fixed and forces are applied at the other end which are equivalent to a single force W passing through the centroid G of the section We take the origin at the centroid of the fixed section, the axis of z along the central line and the axis of x perpendicular to the hypotenuse of the right-angled triangle forming the section. If the axis of y be taken parallel to the hypotenuse, it is found that the axis of x and y are parallel to the principal axes of inertia of the cross sections at their centroids We resolve the force W into two components W_1 and W_2 parallel to the axes of x and y and find the solutions for the two different cases separately The solution when both of them act is to be obtained by combining the two solutions

* *Lond Math Soc Proc* (Series 2), Vol. 20.

Case I.

When W_1 acts along the axis of x and there is no force along OY, we know from Love's Elasticity * that the equilibrium can be maintained if the stresses are such that

$$X_x = Y_y = X_y = 0,$$

$$Z_z = -W_1 (l-z) \frac{x}{I_1},$$

$$X_z = \mu\tau \left(\frac{\partial \phi}{\partial x} - y \right) - \frac{W_1}{2(1+\sigma)I_1} \left\{ \frac{\partial \chi_1}{\partial x} + \frac{1}{2} \sigma x^2 + \left(1 - \frac{\sigma}{2}\right) y^2 \right\},$$

$$Y_z = \mu\tau \left(\frac{\partial \phi}{\partial y} + x \right) - \frac{W}{2(1+\sigma)I_1} \left\{ \frac{\partial \chi_1}{\partial y} + (2+\sigma) xy \right\}$$

where l=length of the beam, I_1, the moment of inertia about the y-axis.

ϕ is the torsion function for the section and χ_1 is a function which satisfies the equation

$$\frac{\partial^2 \chi_1}{\partial x^2} + \frac{\partial^2 \chi_1}{\partial y^2} = 0 \qquad . \quad (1)$$

at all points of the section

$$\text{and } \frac{\partial \chi_1}{\partial \nu} = -\left[\frac{1}{2} \sigma x^2 + \left(1 - \frac{\sigma}{2}\right) y^2 \right] \cos(x, \nu) - (2+\sigma) xy \cos(y, \nu) \qquad (2)$$

at all points of the boundary

τ is a constant of integration which can be determined by making the moment of the stresses about the central line vanish

* Fourth edition, p 332.

The function ϕ for the section was found in a very simple way by C Kolossoff,* who took a different system of axes

With reference to our axes

$$\phi = xy + ay + 9a^2\left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^3 \sinh\frac{2n+1}{2}\pi} \times$$

$$\left[\cosh\frac{2n+1}{6a}\pi(y-x+a)\sin\frac{2n+1}{6a}\pi(y+x-a)\right.$$

$$\left.+\cosh\frac{2n+1}{6a}\pi(y+x-a)\sin\frac{2n+1}{6a}\pi(y-x+a)\right]$$

where a is the perpendicular distance of the hypotenuse from the centroid and the equations of the sides are

$$x=a,$$

$$y=x+2a,$$

$$\text{and}\quad y=-(x+2a)$$

The problem now is to find χ_1 such that

$$\frac{\partial^2\chi_1}{\partial x^2}+\frac{\partial^2\chi_1}{\partial y^2}=0 \qquad \ldots \quad (1)$$

throughout the section

$$\text{and}\quad \frac{\partial\chi_1}{\partial x} = -\left[\frac{\sigma}{2}a^2+\left(1-\frac{\sigma}{2}\right)y^2\right] \text{ when } x=a, \qquad .. \quad (3)$$

$$-\frac{\partial\chi_1}{\partial x}+\frac{\partial\chi_1}{\partial y} = -(1+\sigma)x^2-4a\,\sigma x+2a^2(2-\sigma)$$

$$\text{when } y=x+2a \qquad \ldots \quad (4)$$

$$\text{and}\quad \frac{\partial\chi_1}{\partial x}+\frac{\partial\chi_1}{\partial y} = (1+\sigma)x^2+4a\sigma x-2a^2(2-\sigma)$$

$$\text{when } y=-(x+2a) \quad .. \quad (5)$$

* *Paris C. R* t 178 (1924).

Let us assume

$$\chi_1 = A_1 x + B_1(x^2 - y^2) + C_1\ (x^3 - 3xy^2)$$

$$+ P_1 \frac{108a^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \sinh\ n\pi} \left[\cosh \frac{n\pi}{3a}\ (x+2a)\ \cos \frac{n\pi y}{3a} \right.$$

$$\left. + \cosh \frac{n\pi}{3a}\ y\ \cos \frac{n\pi}{3a}\ (x+2a) \right] \quad . \quad (6)$$

Now since throughout the interval $3a > y > -3a$

$$y^2 - 3a^2 = \frac{36a^2}{\pi^2} \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi y}{3a}$$

we find that all the conditions (1), (3), (4), (5) are satisfied, if

$$\left.\begin{aligned} A_1 &= -2a^2 \\ B_1 &= a\sigma \\ C_1 &= \frac{1+\sigma}{6} \\ P_1 &= \frac{2\sigma - 1}{2}\ . \end{aligned}\right\} \qquad (7)$$

Case II

When the force W_2 acts along the axis of y and there is no force along OX, the requisite stress components are given by the equations *

$$X_z = \mu\tau\left(\frac{\partial \phi}{\partial x} - y\right) - \frac{W_2}{2(1+\sigma)I_2}\left[\frac{\partial \chi_2}{\partial x} + (2+\sigma)xy\right],$$

$$Y_z = \mu\tau\left(\frac{\partial \phi}{\partial y} + x\right) - \frac{W_2}{2(1+\sigma)I_2}\left[\frac{\partial \chi_2}{\partial y} + \tfrac{1}{2}\,\sigma y^2 + \left(1 - \frac{\sigma}{2}\right) x^2\right],$$

$$Z_z = -\frac{W_2}{I_2}\ (l-z)y,$$

$$X_x = Y_y = X_y = 0$$

* Love's Elasticity, 4th Edition, p. 343.

Here I_2 denotes the moment of inertia of the cross section about the x-axis and ϕ has got the same value as before

χ_2 is a plane harmonic function which satisfies the condition

$$\frac{\partial \chi_2}{\partial \nu} = -(2+\sigma)\, xy \cos(x, \nu) - \left[\tfrac{1}{2}\sigma y^2 + \left(1 - \frac{\sigma}{2}\right) x^2\right] \cos(y, \nu)$$

at the boundary

This boundary condition reduces to

$$\frac{\partial \chi_2}{\partial x} = -(2+\sigma)\, ay \text{ when } x = a \tag{8}$$

$$-\frac{\partial \chi_2}{\partial x} + \frac{\partial \chi_2}{\partial y} = (1+\sigma)x^2 + 4ax - 2a^2\sigma \text{ when } y = x + 2a \tag{9}$$

$$\frac{\partial \chi_2}{\partial x} + \frac{\partial \chi_2}{\partial y} = -(1+\sigma)x^2 - 4ax + 2a^2\sigma \text{ when } y = -(x+2a) \tag{10}$$

Let us assume

$$\chi_2 = A_2 x + B_2(x^2 - y^2) + C_2(x^3 - 3xy^2)$$

$$+ P_2 \frac{108a^3}{\pi^3} \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^3 \sinh n\pi} \left[\cosh \frac{n\pi}{3a}(x+2a) \cos \frac{n\pi y}{3a} \right.$$

$$\left. + \cosh \frac{n\pi y}{3a} \cos \frac{n\pi}{3a}(x+2a)\right]$$

$$+ Q_2 \frac{144a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3 \cosh \frac{2n+1}{2}\pi} \times$$

$$\left[\sinh \frac{2n+1}{6a}\pi(x+2a) \sin \frac{2n+1}{6a}\pi y \right.$$

$$\left. + \sinh \frac{2n+1}{6a}\pi y \sin \frac{2n+1}{6a}\pi(x+2a)\right] \quad \dots \tag{11}$$

Now since between the interval $3a>y>-3a$

$$y=\sum_{n=0}^{\infty}(-1)^n\frac{24a}{(2n+1)^2\pi^2}\sin\frac{2n+1}{6a}\pi y,$$ we find that all the conditions (8), (9) and (10) are satisfied, if

$$A_2=2a^2,$$

$$B_2=-a,$$

$$C_2=\frac{1+\sigma}{6},$$

$$P_2=-\frac{1+\sigma}{2},$$

and $Q_2=-(2+\sigma)a$

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4

ON THE INTEGRAL EQUATION OF BESSEL'S FUNCTION $J_n(x)$

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Integral equations corresponding to various well-known linear differential equations of the second order, except the general equation of Bessel, have already been found The object of the present paper is to find an integral equation for Bessel's function $J_n(x)$ directly from the differential equation It is believed that the results obtained are new

I take this opportunity to express my best thanks to Professor Ganesh Prasad for his encouragement and interest

1 Consider the equation

$$X \frac{d^2u}{dX^2} + \frac{du}{dX} + \left(X - \frac{n^2}{X}\right) u = 0 \qquad (1)$$

where u together with its first two derivatives is continuous in the interval (0,1) Equation (1) can evidently be put in the form

$$\frac{d}{dX}\left(X \frac{du}{dX}\right) + \left(X - \frac{n^2}{X}\right) u = 0$$

and is therefore self-adjoint *

Substituting $X = \sqrt{\lambda} x$ in (1) we get

$$\frac{d}{dx}\left(x \frac{du}{dx}\right) - \frac{n^2}{x} u + \lambda x u = 0 \qquad \ldots \qquad (2)$$

or

$$L(u) + \lambda x u = 0 \qquad \ldots \qquad (3)$$

* A R Forsyth, *Theory of Differential Equations*, Pt. III, Vol IV (Cambridge, 1902), p 155.

where

$$L(u) = \frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{n^2}{x} u$$

Let us suppose that the boundary conditions are

$$u(0)=0, \text{ and } u(1)=0$$

2 In order to construct Green's function $K(x,t)$ we observe that (i) K is continuous in the interval $(0,1)$, (ii) $\frac{dK}{dx}$ and $\frac{d^2K}{dx^2}$ are continuous on $(0,t)$, $(t,1)$ separately, and also $L(K)=0$ on $(0,t)$, $(t,1)$ separately

$$(iii) \quad K'(t-0)-K'(t+0)=\frac{1}{t}, \text{ where } K'=\frac{\partial K}{\partial x}$$

Now $L(K)=0$ gives

$$K(x,t)=\begin{cases} A_0 x^n + B_0 x^{-n}, & 0\leq x\leq t \\ A_1 x^n + B_1 x^{-n}, & t\leq x\leq 1. \end{cases}$$

Because of the boundary conditions and the properties of the Green's function we have enunciated above we must have

$$K(x,t)=\begin{cases} -\frac{1}{2n}(t^n-t^{-n})x^n, & 0\leq x\leq t \\ \\ -\frac{1}{2n}t^n(x^n-x^{-n}), & t\leq x\leq 1 \end{cases} \tag{4}$$

3 We can easily show the equivalence of the above boundary problem with a homogeneous integral equation thus:

We observe that

$$L(u)=-\lambda xu$$

and

$$L(K)=0,$$

from which we get

$$uL(K)-KL(u)=\lambda Kxu$$

or

$$\frac{d}{dx}[x(uK'-Ku')]=\lambda Kxu.$$

Integrating the above from 0 to $t-0$, and from $t+0$ to 1, taking account of the restrictions placed on K and u, and also noting that * $R_0(u)=0$, $R_0(K)=0$, $R_1(u)=0$, $R_1(K)=0$, we have

$$u(x)=\lambda \int_0^1 K(x,t)\ t\ u(t)dt, \tag{5}$$

where $K(x,t)$ is given by (4) It may be noted that as usual the equation (5) can easily be converted into one with symmetric kernel on multiplying both the sides by $\sqrt{x}$

4. The most general solution of the equation (1) is given by

$$u=AJ_n(X)+BK_n(X),$$

hence the solution of the equation (2) can be put in the form

$$AJ_n(x\sqrt{\lambda})+BK_n(x\sqrt{\lambda})$$

Now since $u(0)=0$, $B=0$, also because $u(1)=0$, we have

$$J_n(\sqrt{\lambda})=0$$

whose roots will give the characteristic constants These roots are positive and infinite in number The fundamental function corresponding to the root λ_k is given by

$$\phi_k(x)=J_n(x\sqrt{\lambda_k}) \tag{6}$$

5 In order to verify that the equation (5) is the integral equation for $J_n(x\sqrt{\lambda})$, let us substitute the value of $\phi_k(x)$ from (6) for $u(x)$ in (5) Thus

$$J_n(x\sqrt{\lambda_k})=\lambda_k \int_0^1 K(x,t)tJ_n(t\sqrt{\lambda_k})dt$$

$$=-\lambda_k \frac{x^n-x^{-n}}{2n}\int_0^x t^{n+1}J_n(t\sqrt{\lambda_k})dt$$

* Here $\quad R_0(u)\equiv au(0)+b\left[\dfrac{d}{dx}u(x)\right]^{x=0}$

$\qquad R_1(u)\equiv a_1u(1)+b_1\left[\dfrac{d}{dx}u(x)\right]^{x=1}$

and similarly $R_0(K)$ and $R_1(K)$.

$$-\frac{\lambda_k}{2n}x^n\int_x^1 (t^{n+1}-t^{-n+1})J_n(t\sqrt{\lambda_k})\,dt$$

$$=-\frac{\lambda_k}{2n}x^n\int_0^x t^{n+1}J_n(t\sqrt{\lambda_k})\,dt+\frac{\lambda_k}{2n}x^{-n}\int_0^x t^{n+1}J_n(t\sqrt{\lambda_k})\,dt$$

$$-\frac{\lambda_k}{2n}x^n\int_x^1 t^{n+1}J_n(t\sqrt{\lambda_k})\,dt+\frac{\lambda_k}{2n}x^n\int_x^1 t^{-n+1}J_n(t\sqrt{\lambda_k})\,dt$$

$$=-\frac{\lambda_k}{2n}x^n\int_0^1 t^{n+1}J_n(t\sqrt{\lambda_k})\,dt+\frac{\lambda_k}{2n}x^{-n}\int_0^x t^{n+1}J_n(t\sqrt{\lambda_k})\,dt$$

$$+\frac{\lambda_k}{2n}x^n\int_x^1 t^{-n+1}J_n(t\sqrt{\lambda})\,dt$$

Now since*

$$\frac{d[x^nJ_n(x)]}{dx}=x^nJ_{n-1}(x)$$

and

$$\frac{d[x^{-n}J_n(x)]}{dx}=-x^{-n}J_{n+1}(x)$$

we have

$$J_n(x\sqrt{\lambda})=-\frac{\sqrt{\lambda_k}\,x^n}{2n}J_{n+1}(\sqrt{\lambda_k})+\frac{\sqrt{\lambda_k}}{2n}x\,J_{n+1}(x\sqrt{\lambda_k})$$

$$-\frac{\sqrt{\lambda_k}}{2n}x^nJ_{n-1}(\sqrt{\lambda_k})+\frac{\sqrt{\lambda_k}}{2n}x\,J_{n-1}(x\sqrt{\lambda_k})$$

$$=x^nJ_n(\sqrt{\lambda_k})+J_n(x\sqrt{\lambda_k})$$

$$=J_n(x\sqrt{\lambda_k})$$

Thus we have proved that

$$J_n(x\sqrt{\lambda_k})=\lambda_k\int_0^1 K(x,t)\,t\,J_n(t\sqrt{\lambda_k})\,dt$$

* G N. Watson, *Theory of Bessel Functions* (*Cambridge*, 1922), p. 18

ON THE BENDING OF A LOADED ELLIPTIC PLATE

BY

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In a recent paper * Happel has given the solution of the problem of bending of an elliptic plate loaded at the centre The object of the present paper is to obtain the solution of the same problem when the load is concentrated at a focus

1 In a plate slightly bent by transverse forces, the normal displacement w of the middle plane of the plate satisfies the equation†

$$D\nabla_1^4 w = Z' \qquad \dots \quad (1)$$

where Z' is the load per unit of area and D is the flexural rigidity of the plate and is equal to $\frac{2}{3}\frac{Eh^3}{1-\sigma^2}$, $2h$ being the thickness of the plate

At points where there is no load,

$$\nabla_1^4 w = 0 \qquad \dots \quad (2)$$

if we take

$$w = -\frac{W}{8\pi D} r^2 \log r \qquad \dots \quad (3)$$

as a solution of (2), r being the distance of any point from the origin, we have

$$\nabla_1^2 w = -\frac{W}{2\pi D}(\log r + 1)$$

and therefore the normal shearing force N at any point

$$= -D\frac{\partial}{\partial r}\nabla_1^2 w$$

$$= \frac{W}{2\pi r}$$

thus showing that the resultant shearing force on the part of the plate within the circle of radius r is W Hence W is the load at the origin

* *Mathematische Zeitschrift*, Bd , 11 (1921), p 194

† Love, *Theory of Elasticity* (3rd ed), p 492.

2. If we use the transformation

$$x+iy=c\ \cosh\ (\alpha+i\beta)$$

we have

$$\left.\begin{aligned} x&=c\cosh\alpha\cos\beta\ ,\quad y=c\sinh\alpha\sin\beta \\ \text{and}\quad \frac{1}{h^2}&=\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2=\frac{c^2}{2}\ (\cosh 2\alpha-\cos 2\beta) \end{aligned}\right\} \quad .. \quad (4)$$

The curves $\alpha=$constant and $\beta=$constant are a set of confocal ellipses and hyperbolas respectively The foci are given by $\alpha=0$, $\beta=0$, and $\alpha=0$, $\beta=\pi$ If R be the distance of a point (α, β) from the focus $(0, 0)$, then

$$\mathrm{R}=c(\cosh\alpha-\cos\beta) \qquad . \quad (5a)$$

and

$$\log \mathrm{R}=\log\frac{c}{2}+\alpha-2\sum_{n=1}^{\infty}\frac{e^{-n\alpha}}{n}\cos n\beta \qquad . \quad (5b)$$

3 Let there be a load W at the focus (0, 0)

Assume

$$w=-\frac{\mathrm{W}}{8\pi\mathrm{D}}\ \mathrm{R}^2\left(\log\mathrm{R}-\log\frac{c}{2}\right)+w' \qquad (6)$$

where w' satisfies equation (2) and $\nabla^2 w'$ has no singularities within the plate

We have

$$\mathrm{R}^2=\frac{c^2}{2}\ (\cosh 2\alpha+\cos 2\beta+2-4\cosh\alpha\cos\beta).$$

Therefore

$$\begin{aligned} w=-\frac{\mathrm{W}c^2}{16\pi\mathrm{D}}\Bigg[&\left(2+2\alpha+\frac{3}{2}e^{-2\alpha}+\alpha\cosh 2\alpha\right) \\ &-\left(4e^{-\alpha}+e^{\alpha}+\frac{1}{3}e^{-3\alpha}+4\alpha\cosh\alpha\right)\cos\beta \\ &-\left(-\frac{3}{2}-\alpha-\frac{2}{3}e^{-2\alpha}+\frac{1}{12}e^{-4\alpha}\right)\cos 2\beta \\ &-\sum_{n=3}^{\infty}\left\{\frac{2e^{-(n-2)\alpha}}{(n-2)(n-1)n}-\frac{4e^{-n\alpha}}{(n-1)n(n+1)}\right. \\ &\left.+\frac{2e^{-(n+2)\alpha}}{n(n+1)(n+2)}\right\}\cos n\beta\Bigg]+w' \end{aligned} \quad .. \quad (7)$$

where

$$w' = a + a_0 \cosh 2\alpha + (a'_1 \cosh \alpha + a_1 \cosh 3\alpha) \cos \beta$$

$$+ (a_0 + a'_2 \cosh 2\alpha + a_2 \cosh 4\alpha) \cos 2\beta$$

$$+ \sum_{n=3}^{\infty} \{a_{n-2} \cosh (n-2)\alpha + a'_n \cosh n\alpha + a_n \cosh (n+2)\alpha\} \cos n\beta \qquad (8)$$

From (6) and (8) it is easily seen that $\frac{\partial w}{\partial \alpha} = 0$ when $\alpha = 0$

If the edge $\alpha = \alpha_0$ be clamped, we have

$$w = 0 \text{ and } \frac{\partial w}{\partial \alpha} = 0 \qquad (9)$$

when $\alpha = \alpha_0$.

Solving the equations obtained from (9) we have

$$a \sinh 2\alpha_0 = \frac{W}{32\pi D} [3 + 4(1 + \alpha_0) \sinh 2\alpha_0 - 2 \cosh 2\alpha_0 - \cosh^2 2\alpha_0],$$

$$a_0 \sinh 2\alpha_0 = \frac{Wc^2}{32\pi D} [2 - 3e^{-2\alpha_0} + \cosh 2\alpha_0 + 2\alpha_0 \sinh 2\alpha_0],$$

$$a'_1 \cosh^2 \alpha_0 \sinh 2\alpha_0 = -\frac{Wc^2}{64\pi D} [1 + (4e^{-\alpha_0} + e^{\alpha_0})(3 \sinh 3\alpha_0 - \cosh 3\alpha_0) + 4 \cosh \alpha_0 (3\alpha_0 \sinh 3\alpha_0 - \cosh 3\alpha_0) - 4\alpha_0 \sinh \alpha_0 \cosh 3\alpha_0],$$

$$a'_1 \cosh^2 \alpha_0 \sinh 2\alpha_0 = -\frac{Wc^2}{64\pi D} [-3 + 4 \cosh^2 \alpha_0 - e^{-3\alpha_0}(\cosh \alpha_0 + \tfrac{1}{3} \sinh \alpha_0)],$$

$$a'_2 \sinh 2\alpha_0 \cosh 4\alpha_0 = -2a_0 \sinh 4\alpha_0 - \frac{Wc^2}{32\pi D} \left[4\left(-\frac{3}{2} - \alpha_0 - \frac{2}{3} e^{-2\alpha_0} + \frac{1}{12} e^{-4\alpha_0}\right) \sinh 4\alpha_0 - \left(-1 + \frac{4}{3} e^{-2\alpha_0} - \frac{1}{3} e^{-4\alpha_0}\right) \cosh 4\alpha_0\right],$$

$$a_2 \sinh 2\alpha_0 \cosh 4\alpha_0 = a_0 \sinh 2\alpha_0$$

$$-\frac{Wc^2}{32\pi D}\left[-2\left(-\frac{3}{2}-\alpha_0-\frac{2}{3}e^{-2\alpha_0}+\frac{1}{12}e^{-4\alpha_0}\right)\sinh 2\alpha_0\right.$$

$$\left.+\left(-1+\frac{4}{3}e^{-2\alpha_0}-\frac{1}{3}e^{-4\alpha_0}\right)\cosh 2\alpha_0\right],$$

$$a'_n[n \sinh 2\alpha_0 + 2\cosh 2\alpha_0 \sinh (n+2)\alpha_0]$$

$$= -a_{n-2}[n \sinh 4\alpha_0 + 2 \sinh 2n\alpha_0]$$

$$-\frac{Wc^2}{8\pi D}\left[(n+2)\sinh (n+2)\alpha_0\left\{\frac{e^{-(n-2)\alpha_0}}{(n-2)(n-1)n}\right.\right.$$

$$\left.-\frac{2e^{-n\alpha_0}}{(n-1)n(n+1)}+\frac{e^{-(n+2)\alpha_0}}{n(n+1)(n+2)}\right\}$$

$$\left.+\cosh (n+2)\alpha_0\left\{\frac{e^{-(n-2)\alpha_0}}{(n-1)n}-\frac{2e^{-n\alpha_0}}{(n-1)(n+1)}+\frac{e^{-(n+2)\alpha_0}}{n(n+1)}\right\}\right],$$

$$a_n[n \sinh 2\alpha_0 + 2\cosh 2\alpha_0 \sinh (n+2)\alpha_0]$$

$$= a_{n-2}[(n-1)\sinh 2\alpha_0 + \sinh (2n-2)\alpha_0]$$

$$+\frac{Wc^2}{8\pi D}\left[n \sinh n\alpha_0 \left\{\frac{e^{-(n-2)\alpha_0}}{(n-2)(n-1)n}\right.\right.$$

$$\left.-\frac{2e^{-n\alpha_0}}{(n-1)n(n+1)}+\frac{e^{-(n+2)\alpha_0}}{n(n+1)(n+2)}\right\}$$

$$\left.+\cosh n\alpha_0\left\{\frac{e^{-(n-2)\alpha_0}}{(n-1)n}-\frac{2e^{-n\alpha_0}}{(n-1)(n+1)}+\frac{e^{-(n+2)\alpha_0}}{n(n+1)}\right\}\right]$$

In conclusion, I wish to express my thanks to Prof N R. Sen, for his interest in the preparation of the paper

ON THE CONDITION OF ORTHOGONALITY OF TWO CIRCLES

BY

C V H RAO, M A

(University of the Punjab)

1 Given two circles, the Radical Axis is an ancient concept, but the Radical Circle* is of quite recent origin I establish, with cartesians, the chief property of this Circle as a locus, I proceed to point out how the relation between two given circles, their Radical Axis, and the Radical Circle is *independent of metrical considerations* I then indicate how a necessary and sufficient condition could be stated in any system of general homogeneous co-ordinates for two circles to cut orthogonally, and as thus stated it will be apparent that *this condition is purely a condition of projective geometry* I close with a verification that these ideas lead to the usual result in the areal system of co-ordinates

2 Given in cartesians two circles S_1 and S_2 in the canonical form, we define the Radical Circle as the locus of centres of Circles which meet S_1 orthogonally, and are met at the ends of a diameter by S_2 If S_0 is the varying Circle, r_0 its radius, and (x_0, y_0) the centre, the two conditions lead to

$$r_0^2 = S_1(x_0, y_0), \quad -r_0^2 = S_2(x_0, y_0),$$

whence the required locus is $S_1 + S_2 = 0$ and is symmetrical as between the two Circles † The Radical Axis being $S_1 - S_2 = 0$, we may consider the coaxial system of circles fixed by S_1 and S_2 as a linear aggregate wherein the Radical Circle may be defined as the fourth harmonic of the Radical Axis, this last being understood as the one member of the family which degenerates

3 The centre of the Radical Axis, considered as a circle, being an inaccessible point, it follows that the centre of the Radical Circle is

* *Cf* Coolidge —The Circle and the sphere, pp 105-110

† My attention has been drawn to this method of proof by Mr Hans Raj Mittal It may however be said to be contained implicitly in a remark in Coolidge, *loc cit* page 106, end of Theorem 200,

the point midway between the centres of S_1 and S_2, and the Circle itself passes through the two common points of S_1, S_2 But in general it does not pass through the centres of S_1, S_2 If, however, it does pass through the centre of either of the two Circles, so does it pass through the centre of the other as well, and as a consequence the two circles meet orthogonally

Wherefore we say that the necessary and sufficient condition for two circles to intersect orthogonally is that their Radical Circle should pass through the centre of either circle and hence also through the centre of the other

4 Using any system of homogeneous co-ordinates, if $S_1(x, y, z)=0$ and $S_2(x, y, z)=0$ be the two given circles, the necessary and sufficient condition that they should cut at right angles is that $S_1+\lambda S_2=0$ should pass through the centre of either of the two given circles, where the constant λ is to be so chosen that the circle $S_1-\lambda S_2=0$ degenerates into the line at infinity in the particular system of co-ordinates used together with a second line in the finite part of the plane

5 Using areals in particular, consider the circle

$$S(x, y, z)=\sum a x^2+2 fy z=0.$$

It may be expressed in the form

$$\frac{S}{\lambda}=(\sum x)(\sum \frac{\alpha}{\lambda} x)-\sum a^2 yz$$

where

$$\beta+\gamma-2 f=a^2\lambda \quad \text{etc}$$

The Radical Circle of S and the circumcircle of the triangle of reference is then

$$(\sum x)(\sum \frac{\alpha}{\lambda} x)-2\sum a^2 yz=0$$

The condition for orthogonality is that this equation should be satisfied by the co-ordinates of the circumcentre which are proportional to

$$\sin 2A, \sin 2B, \sin 2C$$

We require therefore that

$(\sum \sin 2A)(\sum \alpha \sin 2A)=2\lambda\sum a^2 \sin 2B \sin 2C$, which can be seen to reduce to the condition usually given

Bull Cal Math Soc, Vol XXI, Nos 3 and 4 (1929),

Fig. I

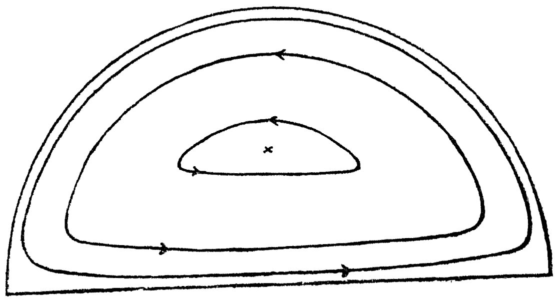
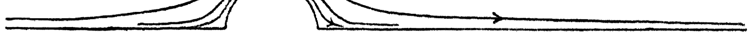


Fig II.



7

ON A CASE OF VORTEX MOTION NEAR SEMICIRCULAR BOUNDARIES AND INFINITE STRAIGHT BOUNDARIES WITH SEMICIRCULAR PROJECTIONS.

BY

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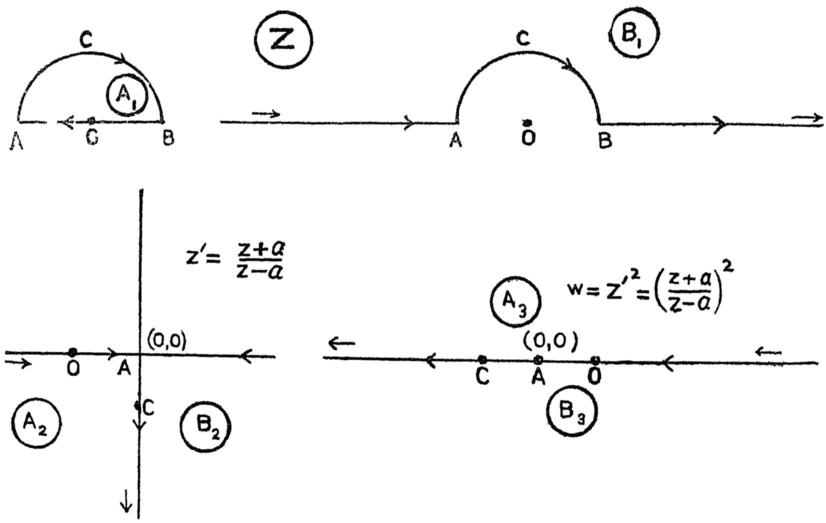
As the resistance offered by an obstacle to a streaming liquid is now definitely ascribed to the existence of vortices behind it, the study of vortex motion near simple geometric boundaries may not be considered quite unfruitful For a complete circular boundary (in two dimensions), a straight vortex is known to describe a concentric circle The method of images which gives the solution easily in this case, is directly inapplicable when the boundary consists of two arcs of circles By conformal transformation however, the region inside or outside the arcs can be converted into the region in an infinite half plane when the treatment of the problem by the method of image immediately leads to the desired solution. A theory of transformation by means of inversion has been given by Routh* but the path of the vortex is fairly complicated and its actual plotting involves some labour In two important cases however, *viz.* when the motion is inside a semicircular boundary or in an infinite straight boundary with semicircular projection, the plotting of the paths can be made to depend on the measurement of certain lengths without any trigonometrical calculations or other additional constructions The object of the note is to draw attention to this point and actually plot the curves in these two cases

* 'Some Applications of conjugate Functions', *Proc L.M.S*, XII, 1881.

Both the cases can be solved by the same formulae of transformation as shown below.



The region inside the semicircular boundary (A_1) is transformed ultimately into the semi-infinite plane on the positive side of the real axis and the infinite straight boundary with semicircular projection (B_1) into the semi infinite plane on the negative side

We have,

$$w=\xi+i\eta=\left(\frac{z+a}{z-a}\right)^2=\frac{(x^2+y^2-a^2)^2-4y^2a^2-4iya(x^2+y^2-a^2)}{\{(x-a)^2+y^2\}^2}$$

so that

$$\xi=\frac{(x^2+y^2-a^2)^2-4y^2a^2}{\{(x-a)^2+y^2\}^2}, \quad \eta=-\frac{4ya(x^2+y^2-a^2)}{\{(x-a)^2+y^2\}^2},$$

and

$$\frac{dw}{dz}=-4a\,\frac{(z+a)}{(z-a)^3},$$

whence

$$\left|\frac{dw}{dz}\right|=4a\,\frac{|z+a|}{|z-a|^3}, \text{ where } |z+a|=\{(x+a)^2+y^2\}^{\frac{1}{2}},$$

$$|z-a|=\{(x-a)^2+y^2\}^{\frac{1}{2}}.$$

Let a vortex at ξ_1, η_1, in the w-plane correspond to a vortex at x_1, y_1 in the z-plane The stream function due to a vortex at ξ_1, η_1 in the semi-infinite plane is given by

$$\psi=\frac{k}{4\pi}\log\frac{(\xi-\xi_1)^2+(\eta-\eta_1)^2}{(\xi-\xi_1)^2+(\eta+\eta_1)^2}\ .$$

Now, the stream function for the motion of a vortex at x_1, y_1 in the z-plane can be found, when the stream function for the motion of the vortex at the corresponding point ξ_1, η_1, in the w-plane is known. If $\chi'(x_1, y_1)$ and $\chi(\xi_1, \eta_1)$ are these two stream functions, we have $\chi'(x_1, y_1)$ given by

$$\chi'(x_1, y_1)=\chi(\xi_1, \eta_1)+\frac{k}{4\pi}\log\left|\frac{dw}{dz}\right|, \dagger$$

Again, to find $\chi(\xi_1, \eta_1)$ we have

$$-\frac{\partial\chi}{\partial\eta_1}=\left(-\frac{\partial\psi'}{\partial\eta}\right)_1, \text{ and } \frac{\partial\chi}{\partial\xi_1}=\left(\frac{\partial\psi'}{\partial\xi}\right)_1,$$

where

$$\psi'=-\frac{k}{4\pi}\log\{(\xi-\xi_1)^2+(\eta+\eta)^2{}_1\}, \text{ and } \left(\frac{\partial\psi'}{\partial\xi}\right)_1, \left(\frac{\partial\psi'}{\partial\eta}\right)_1$$

indicate that in the results of differentiation we are to put ξ_1, η_1 in place of ξ, η

Hence

$$\chi=-\frac{k}{4\pi}\log\eta_1$$

Thus for the region A_1, we have

$$\chi'(x_1, y_1)=-\frac{k}{4\pi}\log\eta_1+\frac{k}{4\pi}\log\left|\frac{dw}{dz}\right|_1$$

$$=\frac{k}{4\pi}\log\frac{|z_1+a|\ |z_1-a|}{y_1(a^2-r_1{}^2)}+\text{const.} \qquad (1)$$

so that the path of P is given by

$$\frac{|z+a|\ |z-a|}{y(a+r)(a-r)}=\text{const}$$

† Routh, *loc. cit.*, p. 83.

In the same way for a vortex at $\xi_2, -\eta_2$ in the w-plane corresponding to one at x_2, y_2 in the z-plane in the region B_1,

$$\chi = -\frac{k}{4\pi} \log (-\eta_2)$$

and

$$\chi'(x_2, y_2) = \frac{k}{4\pi} \log \frac{|z_2+a| \; |z_2-a|}{y_2(r_2+a)(r_2-a)} + \text{const} \qquad . \quad (2)$$

i e , the path of the vortex in the region B_1, is given by

$$\frac{|z+a| \; |z-a|}{y(r+a)(r-a)} = \text{const}$$

In both the cases, to find the constant for a particular point (x_1, y_1) we have only to measure the distances of the point from A and B, and from the origin The path of a vortex can easily be traced In Fig I and Fig II such paths have been shown

The component velocities u_1, v_1 of a vortex at x_1, y_1 can now be found since χ' is the stream function of the vortex, we have

$$u = -\frac{\partial \chi'}{\partial y}, \quad v = \frac{\partial \chi'}{\partial x}$$

Differentiating equation (2) we have in the region II,

$$= -\frac{\partial \chi'}{\partial y} = \frac{k}{4\pi} \left[\frac{1}{y} + \frac{4a^2 y(a^2+r^2-2x^2)}{(r^2-a^2)(r^2+2xa+a^2)(r^2-2xa+a^2)} \right] \quad \dots \quad (3).$$

$$= \frac{\partial \chi'}{\partial x} = -\frac{k}{4\pi} \left[\frac{8a^2 x y^2}{(r^2-a^2)(r^2+2xa+a^2)(r^2-2xa+a^2)} \right] \quad \dots \quad (4)$$

Now, the velocity of a vortex in a semi-infinite plane is $\frac{k}{4\pi} \frac{1}{y}$ in the direction of x-axis. So from equations (3) and (4) we easily see the effect of the semicircular projection.

Considering terms upto the order of a^2/r^2 we find from the above equations the alteration of velocity in the direction of x-axis to be $-\frac{k}{\pi}\left(\frac{a^2}{r^3}\sin\theta\cos 2\theta\right)$ and in the direction of y-axis $-\frac{k}{\pi}\left(\frac{a^2}{r^3}\sin\theta\sin 2\theta\right)$

Hence to the order considered, the effect of the semi-circular projection is to alter the velocity by the amount

$$\frac{k}{\pi}\frac{a^2}{r^3}\sin\theta.$$

Inside the semicircular boundary we write the expressions for velocity components in the form

$$-u=\frac{\partial\chi'}{\partial y}$$

$$=\frac{k}{4\pi}\frac{2y^2(a^4-r^4)+(3y^2-a^2+x^2)\{(x^2-a^2)^2+2y^2(x^2+a^2)+y^4\}}{\{(x+a)^2+y^2\}\{(x-a)^2+y^2\}(a^2-x^2-y^2)y} \qquad (5)$$

$$v=\frac{\partial\chi'}{\partial x}=\frac{k}{4\pi}\frac{8a^2xy^2}{\{(x+a)^2+y^2\}\{(x-a)^2+y^2\}(a^2-x^2-y^2)} \qquad (6)$$

From equation (6), it is seen that $v=0$ when either $y=0$ or $x=0$ The vortex is not to be on the x-axis , so the component vanishes on y-axis

Inside the semicircular boundary we shall have a point where the velocity of the vortex vanishes This we can expect to lie on y-axis , hence putting $x=0$ in the expression for u, we get the equation

$$y^6+5a^2y^4+3a^4y^2-a^6=0$$

as the equation giving y, where the velocity vanishes Now, this equation is a cubic equation in y^2 ; and in graphical representation we may take a as unity, so that we have

$$y^6+5y^4+3y^2-1=0$$

It is seen that this equation has one real root lying between 0 and 1, and this root has been found to be 0 237 very nearly, giving $y = \pm$ 486 The negative value being inadmissible in the case considered we have the null point, *i e*, the point where the vortex remains stand still, given by (0, 0 486) The point is shown in the graph by a cross mark

It is easily seen that in the region **II**, though the y-component of velocity may vanish, there is no such point where the vortex has no velocity at all

The expression for pressure at a point on the boundary is cumbrous and does not appear to be of any use

In conclusion, I wish to express my thanks to Dr N R Sen for suggestions and help

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NOTICE

The Commemoration Volume of the "Bulletin of the Calcutta Mathematical Society" is likely to be out in April, 1930, and will contain the following papers .

(1) J. Larmor (Cambridge) The transmission of free electric waves in the atmosphere, pp. 1-8

(2) H Lamb (Cambridge) On the flow of a compressible fluid past an obstacle, pp 9-16

(3) L Bieberbach (Berlin) Zur Theorie der schlichten Abbildungen, pp 17-20

(4) W Sierpinski (Warswa) A property of ordinal numbers, pp. 21-22

(5) F W Dyson (Greenwich) The variation of latitude, pp 23-30

(6) L Tonelli (Bologna) Sulle Equazioni Funzionale del tipo di Volterra, pp 31-48

(7) L Fejér (Budapest). Uber einen S Bernstein-schen Satz uber die Derivierte eines trigonometrischen Polynoms und uber die Szegosche Verscharfung desselben, pp 49-54

(8) F Riesz (Szeged) Sur l' approximation des fonctions continues et des fonctions sommables, pp. 55-58

(9) T Takagi (Tokyo) On the theory of indeterminate equations of the second degree in two variables, pp 59-66

(10) T Hayashi (Sendai) A problem on probability, pp 67-74

(11) A R Forsyth (London) Geodesic curves in some triple regions within four-dimensional flat space, pp 75 100.

(12) G Prasad (Calcutta) Presidential address, pp. 101-108.

(13) E R Hedrick (Los Angeles) On certain properties of non-analytic functions of a complex variable, pp 109-124

(14) C Caratheodory (Munchen) Bemerkungen zu den Existenz-theoremen der Konformen Abbildung, pp 125-134

(15) D E Smith (New York) Certain questions in the History of mathematics, pp 135-138

(16) N Lusin (Moscow) Sur une propriété des fonctions à carré sommable, pp 139-154

(17) G Prasad (Calcutta) On the function θ in the mean-value theorem of the Differential Calculus, pp 155-184

(18) M Fréchet (Paris) Sur un développement des fonctions abstraites continues, pp 185-192

(19) R Fueter (Zurich) Zur Theorie der Relativ-Abelschen Korper, pp 193-198

(20) E T Whittaker (Edinburgh) : Oliver Heaviside, pp. 199-218

(21) G H Hardy (Oxford) and J E Littlewood (Cambridge) Some problems of Diophantine approximations.

(22) A Sommerfeld (Munchen) Uber die Hauptschnitte eines polydimensionalen Wurfels

(23) H Hahn (Wien) Ueber unendliche Reihen und Absolut-Additive Mengenfunktionen

(24) A N Singh (Lucknow) Some remarks concerning a paper of Dr. Besicovitch

(25) N R Sen and N N Ghosh (Calcutta) Contribution to the theory of gravitational field with axial symmetry

(26) Bibhutibhusan Datta (Calcutta) : On Mahavira's solution of rational triangles and quadrilaterals

(27) E. Salkowski (Charlottenberg) Zur Theorie der Affin-minimal-flachen